Goedel’s Theorem 11
Lecture Contents

- Currys Paradox
- Loebs Theorem
- Loebs Theorem implies the Second Incompleteness Theorem again
- \( \text{Con}_T \) as an undecidable sentence
- Consistent theories that ‘prove’ their own inconsistency
- What’s still to come...
Curry’s paradox

- Consider a sentence $\delta$ such that $\delta \leftrightarrow (\delta \rightarrow \phi)$ is true.
- Then $\phi$ is also true.
- Compare to ‘if I am true then $\phi$’. The Liar is a special case of this.
- If theory $T$ has a standard conditional, and if for every sentence $\phi$ of $T$’s language there is some corresponding sentence $\delta$ such that $T \vdash \delta \leftrightarrow (\delta \rightarrow \phi)$, then $T$ is inconsistent.
An application to set theory

- Naive set theory contains a principle of unrestricted comprehension:

\[
\forall x (x \in \{x \mid \phi\} \iff \phi)
\]

- Set \( \phi \) to be \( x \in x \to \psi \), and then we can derive \( \psi \).
- Russell’s paradox is the special case where \( \psi \) is \( \bot \).
- Compare this to diagonalisation.
A related argument

- $T$ has a truth predicate if we can find some expression $\text{True}$ such that $T \vdash \text{True}[\overline{\phi}] \leftrightarrow \phi$
- If $T$ is consistent, has a standard conditional and knows some arithmetic, it can also have a truth predicate.
- For any $\phi$ we use the diagonalisation lemma to construct a $\delta$ such that $T \vdash \delta \leftrightarrow (\text{True}[\overline{\delta}] \rightarrow \phi)$
- Actually, we do not need a fully fledged truth predicate to get this result.
A weaker truth predicate

Let us write $\top \phi$ as short for some truth predicate $\text{True}^{[\neg \neg \phi]}$.

Suppose the following principles hold for $\top$ for any $\phi$:

1. $T \vdash \top \phi \rightarrow \phi$
2. $T \vdash \top (\phi \rightarrow \psi) \rightarrow (\top \phi \rightarrow \top \psi)$
3. $T \vdash \top \phi \rightarrow \top \top \phi$
4. $T \vdash \phi$ implies $T \vdash \top \phi$

These are enough to derive $\phi$ given the diagonalisation lemma:

$$T \vdash \delta \leftrightarrow (\top \delta \rightarrow \phi)$$
Loeb’s theorem

- Recall the derivability conditions
  1. If $T \vdash \phi$ then $T \vdash \Box \phi$
  2. $T \vdash \Box(\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)$
  3. $T \vdash \Box \phi \rightarrow \Box \Box \phi$

- If $T$ knows some arithmetic, and the derivability conditions hold, then if $T \vdash \Box \phi \rightarrow \phi$ then $T \vdash \phi$.

- Again, consider $\delta$ such that $T \vdash \delta \leftrightarrow \Box \delta \rightarrow \phi$

- Corollary: the sentence ‘I am provable’: $H \leftrightarrow \text{Prov}[\overline{H}]$ is provable.

- Corollary: the second incompleteness theorem, for $T \vdash \neg \Box \bot \leftrightarrow \Box \bot \rightarrow \bot$. 

$G_T$ and $\text{Con}_T$ are provably equivalent in $T$

- Suppose $T$ is sufficiently strong so that the derivability conditions hold (e.g. $T$ is $\Sigma$-normal).
- For any sentence $\phi$, $T \vdash \neg \square \phi \rightarrow \text{Con}_T$.
- So $T$ knows nothing about what it cannot prove.
- We already know that $T \vdash \text{Con}_T \rightarrow \neg \square G_T$.
- If $T$ satisfies the derivability conditions then $T \vdash \text{Con}_T \leftrightarrow G_T$.
- So $\text{Con}_T$ is undecidable in $T$, and is not self-referential.
A theory that ‘proves’ its own inconsistency

- If $T'$ is stronger than $T$, then if $T$ is inconsistent then so is $T$.
- A sufficiently strong theory, e.g. a $\Sigma$-normal one, will be able to derive that $\neg \text{Con}_T \rightarrow \neg \text{Con}_{T'}$.
- Take $PA$ and form $PA'$ by adding $\neg \text{Con}_{PA}$ as a new axiom.
- Since $PA$ is $\Sigma$-normal one, then $PA \vdash \text{Con}_{PA} \rightarrow \text{Con}_{PA'}$.
- So $PA' \vdash \neg \text{Con}_{PA'}$ and is $\omega$-inconsistent.
Further material

- No consistent, sufficiently strong, axiomatised formal theory is decidable.
- A consistent, sufficiently strong, axiomatized formal theory cannot be negation complete.
- We have been working with p.r. decidability which is not quite ‘sufficiently Strong’.
- We extend the idea of a primitive recursive function to a \textit{recursive} function.
- Church’s thesis is the idea that all effectively computable functions are recursive.
Recursive Functions

- Recursive functions are defined just like p.r. ones with an additional clause.
- If $g$ is recursive and $\forall \vec{x} \exists z g(\vec{x}, z) = 0$ $f$ is defined as:
  \[
  f(\vec{x}, \vec{y}) = \mu z g(\vec{x}, z) = 0
  \]
  then $f$ is recursive.
- $Q$ can functionally capture all the recursive functions and all our previous results carry through.
- We also conclude that the property of being the g.n. of a recursive function is not itself recursive.
Turing Computability

- We could have done the whole course in terms of computers and computability.
- The Turing-computable functions are just the recursive functions.
- Furthermore we get a negative solution to the halting problem, which is ultimately equivalent to the result above.