Gödel Without Tears – 4

First-order Peano Arithmetic

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Here's the story so far. We noted in Episode 1 that Gödel showed, more or less,

Theorem 1. *If T is a sound formalized theory whose language contains the language of basic arithmetic, then there will be a true sentence G_T of basic arithmetic such that $T \nvdash \mathsf{G}_T$ and $T \nvdash \neg\mathsf{G}_T$, so T must be negation incomplete.*

Of course, we didn't *prove* that theorem, though we waved an arm airily at the basic trick that Gödel uses to establish the theorem – namely we 'arithmetize syntax' (i.e. numerically code up facts about provability in formal theorems) and then construct a Gödel sentence that sort-of-says 'I am not provable'.

We did note, however, that this theorem invokes the assumption that we dealing with a *sound* theory, and of course soundness is a *semantic* notion. For various reasons, Gödel thought it essential to establish that we can get incompleteness making merely syntactic assumptions, thus:

Theorem 2. *For any consistent formalized theory T which contains a certain modest amount of arithmetic (and has a certain additional desirable property that any sensible formalized arithmetic will share), there is a sentence of basic arithmetic G_T such that $T \nvdash \mathsf{G}_T$ and $T \nvdash \neg\mathsf{G}_T$, so T must be negation incomplete.*

Here, 'contains enough arithmetic' means proving enough (a syntactically characterizable condition). This theorem with syntactic assumptions is the sort of thing that's usually referred to as The First Incompleteness Theorem, and of course we again didn't prove it. Indeed, we didn't even say what that 'modest amount of arithmetic' is (nor did we say anything about that 'additional desirable property'). So Episode 1 was little more than a gesture in the right direction.

In Episode 2, we did a bit better, in the sense that we actually gave a *proof* of the following theorem:

Theorem 6. *A consistent, sufficiently strong, axiomatized formal theory cannot be negation complete.*

The argument was nice, as it shows that we can get incompleteness results without calling on the arithmetization of syntax and the construction of Gödel sentences. However the argument depended on working with an informal, intuitive, notion of 'decidable property'. And, as we noted, the result is weaker than Theorem 2 for it doesn't tell us that there will be a formally undecidable *arithmetic* sentence. Moreover, the discussion in Episode 2 doesn't give us much clue what a 'sufficiently strong' theory might look like.

Episode 3 took a step towards filling that last gap (and also towards telling us what the 'modest amount of arithmetic' mentioned in Theorem 2 amounts to). We first looked at BA, the quantifier-free arithmetic of the addition and multiplication of particular numbers. This is a complete (and hence decidable!) theory – but of course it is only complete, i.e. able to decide every sentence constructible in its language, because its language is indeed so weak.

If we augment the language of BA by allowing ourselves the usual apparatus of first-order quantification, and replace the schematically presented axioms of BA with their obvious universally quantified correlates (and add in the axiom that every number bar zero is a successor) we get Robinson Arithmetic Q. Since we've added pretty minimally to what is given in the axioms of BA while considerably enriching its language, it is no surprise that we have

Theorem 13. Q *is negation-incomplete.*

And we can prove this without any fancy Gödelian considerations. A familiar and simple kind of model-theoretic argument is enough to do the trick: we find a deviant interpretation of Q's syntax which is such as to make the axioms all true but on which $\forall x(0 + x = x)$ is false, thus establishing $Q \nvdash \forall x(0 + x = x)$. And since Q is sound on the built-in interpretation of its language, we also have $Q \nvdash \neg\forall x(0 + x = x)$.

Q, then, is a very weak arithmetic. Still, it will turn out to be the 'modest amount of arithmetic' needed to get Theorem 2 to fly. Also a theory's containing Q makes it a 'sufficiently strong' theory in the sense of Theorem 6. Of course establishing *these* facts is a non-trivial task for later: but they do explain why Q is so interesting despite its weakness.

Now read on ...

11 Arithmetical Induction

For a moment, put $\varphi(x)$ for $(0 + x = x)$. Then as we noted, for any particular n, $Q \vdash \varphi(n)$. But we showed that $Q \nvdash \forall x\varphi(x)$. In other words, Q can separately prove all instances of $\varphi(n)$ but can't prove the corresponding simple generalization. So let's consider what proof-principle we might add to Q to fill this sort of gap.

11.1 The ω-rule

Q, to repeat, proves each of $\varphi(0), \varphi(1), \varphi(2), \varphi(3), \ldots$. Suppose then that we added to Q the rule that we can infer as follows:

$$\frac{\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \\ \varphi(0) & \varphi(1) & \varphi(2) & \varphi(3) & \vdots \end{array}}{\forall x\varphi(x)}$$

This rule – or rather the generalized version for any φ – is what's called *the ω-rule*. It is evidently a sound rule: if each $\varphi(n)$ is true then indeed all numbers n satisfy $\varphi(x)$. Adding the ω-rule would certainly repair the gap we exposed in Q. But of course there's a snag: the ω-rule is infinitary. It takes as input an infinite number of premisses. So proofs invoking this ω-rule will be infinite arrays. And being infinite, they cannot be mechanically checked in a finite number of steps to be constructed according to our expanded rules. In sum, then, a theory with a proof-system that includes an infinitary ω-rule can't count as a formalized theory according to Defn. 3.

There is some technical interest in investigating infinitary logics which allow infinitely long sentences (e.g. infinite conjunctions) and/or infinite-array proofs. But there is a clear sense in which such logics are not of practical use, and cannot be used to regiment how we in fact argue. The finiteness requirement we impose on formalized theories is, for that reason, not arbitrary. And so we'll stick to that requirement, and hence have to ban ω-rule.

11.2 Replacing an infinitary rule with a finite one

To repeat, as well as proving $\varphi(0)$, Q also proves $\varphi(1), \varphi(2), \varphi(3), \ldots$. And it isn't, so to speak, a global accident that Q can prove all those. Rather, Q proves them in a uniform way.

To bring this out, note that we have the following proof in Q:

1.	$\quad\varphi(n)$	Supposition
2.	$\quad 0 + n = n$	Unpacking the definition
3.	$\quad S(0 + n) = Sn$	From 2 by LL
4.	$\quad (0 + Sn) = S(0 + n)$	Instance of Axiom 5
5.	$\quad (0 + Sn) = Sn$	From 3, 4
6.	$\quad\varphi(Sn)$	Applying the definition
7.	$(\varphi(n) \to \varphi(Sn))$	From 1, 6 by Conditional Proof
8.	$\forall x(\varphi(x) \to \varphi(Sx))$	From 7, since n was arbitrary.

Given Q trivially proves $\varphi(\mathsf{0})$, we can appeal to $\forall \mathsf{x}(\varphi(\mathsf{x}) \to \varphi(\mathsf{Sx}))$ to derive $\varphi(\mathsf{0}) \to \varphi(\mathsf{1})$, and so modus ponens gives us $\varphi(\mathsf{1})$. The same generalization also gives us $\varphi(\mathsf{1}) \to \varphi(\mathsf{2})$, so another modus ponens gives us $\varphi(\mathsf{2})$. Now we can appeal to our generalization again to get $\varphi(\mathsf{2}) \to \varphi(\mathsf{3})$, and so can derive $\varphi(\mathsf{3})$. Keep on going! In this way, $\varphi(\mathsf{0})$ and $\forall \mathsf{x}(\varphi(\mathsf{x}) \to \varphi(\mathsf{Sx}))$ together prove all of $\varphi(\mathsf{0}), \varphi(\mathsf{1}), \varphi(\mathsf{2}), \varphi(\mathsf{3}), \ldots$, which in turn, by the sound ω-rule, entail $\forall \mathsf{x}\varphi(\mathsf{x})$.

But now we can evidently cut out the infinity of intermediate steps in that last bit of motivating argument, and that will leave us a nice *finitary* rule

$$\frac{\varphi(\mathsf{0}) \qquad \forall \mathsf{x}(\varphi(\mathsf{x}) \to \varphi(\mathsf{Sx}))}{\forall \mathsf{x}\varphi(\mathsf{x})}$$

This is again evidently sound whatever predicate we put for $\varphi(\xi)$ which expresses a genuine numerical property. Add this finitary rule to Q and we'll evidently at least patch the gap we found.

11.3 Induction: the basic idea

The basic idea reflected in that formal rule is as follows.

> Whatever numerical property we take, if we can show that (i) zero has that property, and also show that (ii) this property is always passed down from a number n to the next Sn, then this is enough to show (iii) the property is passed down to *all* numbers.

This is the *principle of arithmetical induction*, and is a standard method of proof for establishing arithmetical generalizations.

Here's a little example of the principle at work in an everyday informal mathematical context. Suppose we want to establish that the sum of the first n numbers is $n(n+1)/2$. Well, define $\psi(n)$ to hold if that claim is correct for n. Then (i) trivially the result holds for the sum of zero numbers is zero, so $\psi(0)$ is true! And (ii) suppose now the claim holds for a particular n. Then the sum of the first $n+1$ numbers is $n(n+1)/2 + (n+1) = (n+1)(n+2)/2 = (Sn)(Sn+1)/2$. Which means that $\psi(Sn)$ will be true too. So (i) $\psi(0)$, and (generalizing) (ii) for all numbers n, if $\psi(n)$, then $\psi(Sn)$. Therefore, as we want, by induction the claim holds for all numbers.

Here's another example of same principle at work, in telegraphic form. Suppose we want to show that all the theorems of a certain Hilbert-style axiomatized propositional calculus are tautologies. Define $\chi(n)$ to be true if the conclusions of proofs up to n steps long are tautologies. Then we show that $\chi(0)$ is true (trivial!), and then argue that if $\chi(n)$ then $\chi(Sn)$ (e.g. we note that the last step of an $n+1$ step must either be an instance of an axiom, or follow by modus ponens from two earlier conclusions which – since $\chi(n)$ is true – must themselves be tautologies, and either way we get another tautology). Then 'by induction on the length of proofs' we get the desired result.

11.4 A word to worried philosophers

Beginning philosophers, in week one of their first year logic course, have the contrast between deductive and inductive arguments dinned into them. So emphatically is the distinction made, so firmly are they drilled to distinguish conclusive deductive argument from merely probabilistic inductions, that some can't help feeling initially uncomfortable when they first hear of 'induction' being used in arithmetic!

So let's be clear. We have a case of empirical, non-conclusive, induction, when we start from facts about a limited sample and infer a claim about the whole population. Number off the swans, for example, and let $\varphi(n)$ say that swan $\#n$ is white. We sample some swans and run (say) k checks showing that $\varphi(0)$, $\varphi(1)$, $\varphi(2)$, ..., $\varphi(k)$ are all true. We hope that these are enough to be representative of the whole population of swans, and so – taking a chance – infer that for all n, $\varphi(n)$, now quantifying over over all (numbers for) swans, jumping beyond the sample of size k. The gap between the sample and the whole population, between the particular bits of evidence and the universal conclusion, allows space for error. The inference isn't deductively watertight.

By constrast, in the case of arithmetical induction, we start not from a bunch of claims about particular numbers but from an already universally quantified claim about all numbers, i.e. $\forall x(\varphi(x) \to \varphi(Sx))$. We put that universal claim together with the particular claim $\varphi(0)$ to derive another universal claim, $\forall x\varphi(x)$. This time, then, we are going from universal to universal, and there is no deductive gap.

You might say 'Pity, then, that we use the same word in talking of empirical induction and arithmetical induction when they are such different kinds of inference.' True.

11.5 The induction axiom, the induction rule, the induction schema

The basic idea, we said in §11.3, is that for any property of numbers, if zero has it and it is passed from one number to the next, then all numbers have it. This intuitive principle is a generalization over properties of numbers. Hence to frame a corresponding formal version, it seems that we should ideally use a language that enables us to generalize not just over numbers but over properties of numbers. In a phrase, we'd ideally need to be working in a *second-order* theory, which allows second order quantifiers – i.e. we have not only first-order quantifiers running over numbers, but also a further sort of quantifier which runs over arbitrary-properties-of-numbers. Despite that, however, for now we'll concentrate on formal theories whose logical apparatus involves only regular first-order quantification. Note: this isn't down to some perverse desire to fight with one hand tied behind our backs: there are some troublesome issues about second-order logic, about which more perhaps anon here – and certainly more in your advanced logic course.

But if we don't have second-order quantifiers available to range over properties of numbers, how can we handle induction? Well, one way is to adopt the *induction rule* we encountered in §11.2. So long as $\varphi(x)$ expresses a kosher property – and we'll say in a moment what that might come to – we can apply the inference rule

$$\frac{\varphi(0) \qquad \forall x(\varphi(x) \to \varphi(Sx))}{\forall x\varphi(x)}$$

Alternatively, we could say that for every kosher $\varphi(x)$, the corresponding instance of the *induction schema*

$$[\varphi(0) \land \forall x(\varphi(x) \to \varphi(Sx))] \to \forall x\varphi(x)$$

is to be an axiom. Evidently, having the rule and having all instances of the schema comes to just the same.

Techie note. Strictly speaking, we'll also want to allow uses of the inference rule where φ has slots for additional variables dangling free. Equivalently, we will take the axioms to be the universal closures of instances of the induction schema with free variables. For more explanation, see *IGT*, §10.2, and see the idea being put to work in *IGT*, §10.3. We won't fuss about elaborating this point here.

12 First-order Peano Arithmetic

12.1 Getting generous with induction

Suppose then we start again from Q, and aim to build a richer theory in the language L_A (as defined in §10.1) by adding induction.

Any instance of the induction schema, we said, should be intuitively acceptable as an axiom, so long as we replace φ in the schema by a suitable open wff which expresses a genuine property/relation. Well, consider *any* open wff φ of L_A. This will be built from no more than the constant term '0', the familiar successor, addition and multiplication functions, plus identity and other logical apparatus. Therefore – you might very well suppose – it ought also to express a perfectly determinate arithmetical property or relation (even if, in the general case, we can't always decide whether a given number n has the property or not). *So why not be generous and allow any open L_A wff at all to be substituted for φ in the schema?*

Here's a positive argument for generosity. Remember that instances of the induction schema (for monadic predicates) are *conditionals* which look like this:

$$(\{\varphi(0) \ \land \ \forall x(\varphi(x) \to \varphi(Sx))\} \ \to \ \forall x\varphi(x))$$

So they actually only allow us to derive some $\forall x\varphi(x)$ when we can *already* prove the corresponding (i) $\varphi(0)$ and also can prove (ii) $\forall x(\varphi(x) \to \varphi(Sx))$. But if we can already prove things like (i) and (ii) then aren't we already committed to treating φ as a respectable predicate? For given (i) and (ii), we can already prove each and every one of $\varphi(0)$, $\varphi(S0)$, $\varphi(SS0)$, ... However, there are no 'stray' numbers which aren't denoted by some numeral; so that means (iv) that we can prove of each and every number that φ is true of it. What more can it possibly take for φ to express a genuine property that indeed holds for every number, so that (v) $\forall x\varphi(x)$ is true? In sum, it seems that we can't overshoot by allowing instances of the induction schema for *any* open wff φ of L_A with one free variable. The only *usable* instances from our generous range of axioms will be those where we can prove the antecedents (i) and (ii) of the relevant conditionals: and in those cases, we'll have every reason to accept the consequents (v) too. (Techie note: the argument generalizes in the obvious way to the case where $\varphi(x)$ has other variables dangling free.)

12.2 Introducing PA

Suppose then that we accept the conclusion of our last argument, and now take it that *any* open wff of L_A can be used in the induction schema. This means moving on from Q, and jumping right over a range of possible intermediate theories, to adopt the much richer theory of arithmetic that we can briskly define as follows:

Defn. 20. PA – First-order Peano Arithmetic[1] – *is the first-order theory whose language is* L_A *and whose axioms are those of* Q *plus the [universal closures of]* all *instances of the induction schema that can be constructed from open wffs of* L_A.

Plainly, it is still decidable whether any given wff has the right shape to be one of the new axioms, so this is a legitimate formalized theory.

Given its very natural motivation, PA is the benchmark axiomatized first-order theory of basic arithmetic. Just for neatness, then, let's bring together all the elements of its specification in one place. But first, a quick observation. PA allows, in particular, induction for the formula

$$\varphi(x) =_{\text{def}} (x \neq 0 \;\to\; \exists y(x = Sy)).$$

But now note that the corresponding $\varphi(0)$ is a trivial logical theorem. Likewise, $\forall x\varphi(Sx)$ is an equally trivial theorem, and that entails $\forall x(\varphi(x) \to \varphi(Sx))$. So we can use an instance of the Induction Schema inside PA to derive $\forall x\varphi(x)$. But that's just Axiom 3 of Q. So our initial presentation of PA – as explicitly having all the Axioms of Q plus the instances of the Induction Schema – involves a certain redundancy. Bearing that in mind, here's our ...

12.3 Summary overview of PA

First, to repeat, the *language* of PA is L_A, a first-order language whose non-logical vocabulary comprises just the constant '0', the one-place function symbol 'S', and the two-place function symbols '+', '×', and whose intended interpretation is the obvious one.

Second, PA's deductive *proof system* is some standard version of classical first-order logic with identity. The differences between various presentations of first-order logic of course don't make a difference to what sentences can be proved in PA. It's convenient, however, to fix officially on a Hilbert-style axiomatic system for later metalogical work theorizing about the theory.

And third, its non-logical *axioms* – eliminating the redundancy from our original listing and renumbering – are the following sentences:

Axiom 1. $\forall x(0 \neq Sx)$

Axiom 2. $\forall x\forall y(Sx = Sy \to x = y)$

Axiom 3. $\forall x(x + 0 = x)$

[1] The name is conventional. Giuseppe Peano did publish a list of axioms for arithmetic in 1889. But they weren't first-order, only explicitly governed the successor relation, and – as he acknowledged – had already been stated by Richard Dedekind.

Axiom 4. $\forall x \forall y (x + Sy = S(x + y))$

Axiom 5. $\forall x (x \times 0 = 0)$

Axiom 6. $\forall x \forall y (x \times Sy = (x \times y) + x)$

plus every sentence that is the universal closure of an instance of the following

Induction Schema $(\{\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(Sx))\} \to \forall x \varphi(x))$ *where* $\varphi(x)$ *is an open wff that has '*x*', and perhaps other variables, free.*

12.4 What PA can prove

PA proves $\forall x(x \neq Sx)$. Just take $\varphi(x)$ to be $x \neq Sx$. Then PA trivially proves $\varphi(0)$ because that's Axiom 1. And PA also proves $\forall x(\varphi(x) \to \varphi(Sx))$ by contraposing Axiom 2. And then an induction axiom tells us that if we have both $\varphi(0)$ and $\forall x(\varphi(x) \to \varphi(Sx))$ we can deduce $\forall x \varphi(x)$, i.e. no number is a self-successor. It's as simple as that. Yet this trivial little result is worth noting when we recall our deviant interpretation which makes the axioms of Q true while making $\forall x(0 + x = x)$ false: that had Kurt Gödel himself added to the domain as a rogue self-successor. A bit of induction, however, rules out self-successors.

And so it goes: the familiar basic truths about elementary general truths about the successor function, addition, multiplication and ordering (with the order relation as defined in §10.4) are all provable in PA using induction (and rule out other simple deviant models) There are more than enough examples worked through in *IGT*, which we won't repeat here! So we might reasonably have hoped – at least before we'd heard of Gödel's incompleteness theorems – that PA would turn out to be a complete theory that indeed pins down all the truths of L_A.

Here's another fact that might well have encouraged this hope, pre-Gödel. Suppose we define the language L_P to be L_A without the multiplication sign. Take P to be the theory couched in the language L_P, whose axioms are Q's now familiar axioms for successor and addition, plus the universal closures of all instances of the induction schema that can be formed in the language L_P. In short, P is PA minus multiplication. *Then* P *is a negation-complete theory of successor and addition.* We are not going to be able to prove that – the argument uses a standard model-theoretic method called 'elimination of quantifiers' which isn't hard, but it would just take too long to explain. Note, though, that the availability of a complete formalized theory for successor and addition was proved as early as 1929 by Mojżesz Presburger, then a young graduate student.

So the situation is as follows, and was known before Gödel got to work. (i) There is a complete formalized theory BA whose theorems are exactly the quantifier-free truths expressible using successor, addition and multiplication (and the connectives). (ii) There is another complete formalized theory (equivalent to PA minus multiplication) whose theorems are exactly the first-order truths expressible using just successor and addition. Against this background, Gödel's result that adding multiplication in order to get full PA gives us a theory which is incomplete and incompletable (if consistent) comes as a rather nasty surprise. It certainly wasn't obviously predictable that multiplication would make all the difference. Yet it does. In fact, as we've said before, as soon we have an arithmetic as strong as Q, we get incompleteness.

And by the way, it isn't that a theory of multiplication must in itself be incomplete. In 1929, Thoralf Skolem showed that there is a complete theory for the truths expressible in a suitable first-order language with multiplication but lacking addition or the successor function. So why then does putting multiplication together with addition and successor produce incompleteness? The answer will emerge shortly enough, but pivots on the fact that an arithmetic with all three functions built in can express/capture *all* 'primitive recursive' functions. But we'll have to wait to the next Episode to explain what that means.

13 Quantifier complexity and weaker arithmetics

PA is the canonical, most natural, first-order theory of the arithmetic of successor, addition and multiplication. Indeed it is arguable that *any* proposition about successor, addition and multiplication that can be seen to be true just on the basis of our grasp of the structure of the natural numbers can be shown to be true in PA (for discussion of this, see *IGT*, §23.3). Still there is some formal interest in exploring weaker systems, sitting between Q and PA, systems which have *some* induction, but not induction for all open L_A wffs. For example, there is some interest in the theories you get by allowing as axioms instances of the induction schema induction for so-called Δ_0 wffs, or so-called Σ_1 wffs. Now, we are *not* going to explore these weak arithmetics here. But, irrespective of that, it is in fact worth knowing what Δ_0 , Σ_1, and Π_1 wffs are. So this section briskly explains.

13.1 Bounded quantification

As we said before in §10.5, we often want to say that all/some numbers less than or equal to a given number have some particular property. We can now express such claims in formal arithmetics like Q and PA using wffs of the shape $\forall\xi(\xi \leq \kappa \to \varphi(\xi))$ and $\exists\xi(\xi \leq \kappa \wedge \varphi(\xi))$, where $\xi \leq \zeta$ is just short for $\exists \mathsf{v}(\mathsf{v} + \xi = \zeta)$. And it is standard to further abbreviate such wffs by $(\forall\xi \leq \kappa)\varphi(\xi)$ and $(\exists\xi \leq \kappa)\varphi(\xi)$ respectively.

For any theory T containing Q – and hence for T = PA in particular – we have results like these:

1. For any n, $T \vdash \forall \mathsf{x}(\{\mathsf{x} = \mathsf{0} \vee \mathsf{x} = \mathsf{1} \vee \ldots \vee \mathsf{x} = \overline{\mathsf{n}}\} \to \mathsf{x} \leq \overline{\mathsf{n}})$.
2. For any n, $T \vdash \forall \mathsf{x}(\mathsf{x} \leq \overline{\mathsf{n}} \to \{\mathsf{x} = \mathsf{0} \vee \mathsf{x} = \mathsf{1} \vee \ldots \vee \mathsf{x} = \overline{\mathsf{n}}\})$.
3. For any n, if $T \vdash \varphi(\mathsf{0})$, $T \vdash \varphi(\mathsf{1})$, …, $T \vdash \varphi(\overline{\mathsf{n}})$, then $T \vdash (\forall \mathsf{x} \leq \overline{\mathsf{n}})\varphi(\mathsf{x})$.
4. For any n, if $T \vdash \varphi(\mathsf{0})$, or $T \vdash \varphi(\mathsf{1})$, …, or $T \vdash \varphi(\overline{\mathsf{n}})$, then $T \vdash (\exists \mathsf{x} \leq \overline{\mathsf{n}})\varphi(\mathsf{x})$.

In other words, theories like Q and PA 'know' that bounded universal quantifications behave like finite conjunctions and bounded existential quantifications behave like finite disjunctions. Hold on to that thought!

13.2 Δ_0 wffs

Let's informally say that

Defn. 21. *An L_A wff is Δ_0 if its only quantifications are bounded ones.*

For a fancied-up definition, see *IGT*, §9.5. So a Δ_0 wff is one which is built up using the successor, addition, and multiplication functions, identity, the less-than-or-equal-to relation (defined as usual), plus the familiar propositional connectives and/or *bounded* quantifications. Since we know from Theorem 11 that even Q can correctly decide all quantifier-free L_A sentences, and Q knows that *bounded* quantifications behave just like conjunctions/disjunctions, it won't be a surprise to hear that we have

Theorem 15. Q *(and hence* PA*) can correctly decide all Δ_0 sentences.*

13.3 Σ_1 and Π_1 wffs

We now say, again informally, that

Defn. 22. *An L_A wff is Σ_1 if it is (or is equivalent to) a Δ_0 wff preceded by one or more unbounded* existential *quantifiers. And a wff is Π_1 if it is (or is equivalent to) a Δ_0 wff preceded by one or more unbounded* universal *quantifiers.*

As a memnonic, it is worth remarking that 'Σ' in the standard label 'Σ_1' comes from an old alternative symbol for the existential quantifier, as in ΣxFx – that's a Greek 'S' for '(logical) sum'. Likewise the 'Π' in 'Π_1' comes from corresponding symbol for the universal quantifier, as in ΠxFx – that's a Greek 'P' for '(logical) product'. And the subscript '1' in 'Σ_1' and 'Π_1' indicates that we are dealing with wffs which start with *one* block of similar quantifiers, respectively existential quantifiers and universal quantifiers. By the same token, a Π_2 wff is one that starts with *two* blocks of quantifiers, a block of universal quantifiers followed by a block of existential quantifiers followed by a bounded kernel. And so it goes.

13.4 Two results

Here's another pretty trivial result:

Theorem 16. Q *(and hence* PA*) can prove any true* Σ_1 *sentences (is '*Σ_1*-complete').*

Proof. Take, for example, a sentence of the type $\exists x \exists y \varphi(x, y)$, where $\varphi(x, y)$ is Δ_0. If this sentence is true, then for some pair of numbers m, n, the Δ_0 sentence $\varphi(\overline{m}, \overline{n})$ must be true. But then by Theorem 15, Q proves $\varphi(\overline{m}, \overline{n})$ and hence $\exists x \exists y \varphi(x, y)$, by existential introduction.

Evidently the argument generalizes for any number of initial quantifiers, which shows that Q proves all truths which are (or are equivalent to) some Δ_0 wff preceded by one or more unbounded existential quantifiers. □

But if that's trivial, the following consequence is more fun:

Theorem 17. *If* T *is a consistent theory which includes* Q*, then every* Π_1 *sentence it proves is true.*

Proof. Suppose T proves a *false* Π_1 sentence φ. Then $\neg\varphi$ will be a *true* Σ_1 sentence. But in that case, since T includes Q and so is 'Σ_1-complete', T will prove $\neg\varphi$, making T inconsistent. Contraposing, if T is consistent, any Π_1 sentence it proves is true. □

This is, in its way, a rather remarkable observation. It means that we don't have to fully *believe* a theory T – i.e. don't have to accept *all* its theorems are true on the interpretation built into T's language – in order to use it to establish that some Π_1 arithmetic generalization is true. For example, it turns out that, with some trickery, we can state for example Fermat's Last Theorem as a Π_1 sentence. And Andrew Wiles showed how to prove Fermat's Last Theorem using some seriously heavy-duty infinitary mathematics. Now we see, intriguingly, that we don't have to believe that infinitary mathematics is true – whatever that means when things get wildly infinitary! – but only *consistent*, to take him as establishing that the Π_1 arithmetical claim which is the Theorem is true.

Now read *IGT*, chs. 9 and 10.