Modal Logic 1

Lecture Contents

- Propositional modal Languages
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- The notion of a structure and a valuation
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Propositional logic

- A simple language of propositional logic:

$$A ::= p \mid \neg A \mid B \to C \mid B \wedge C \mid B \vee C$$

- We could, of course, make do with just implication and negation, or even implication and absurdity: $\bot$.

Propositional modal logic

- A simple language of propositional modal logic:

$$A ::= p \mid \neg A \mid B \to C \mid B \land C \mid B \lor C \mid \Box A \mid \Diamond A$$

- Again, we could make do with only $\Box$ and define:

$$\Diamond A \equiv \neg \Box \neg A$$

Interpretations of the box operator

$\Box A$	$\Diamond A$
It is necessary that A	
It will always be that A	
It is known that A	
It is obligatory that A	
A is provable in PA	
The program terminates with A	
A is a rational belief	

Interpretations of the box operator

$\Box A$	$\Diamond A$
It is necessary that A	It is possible that A
It will always be that A	
It is known that A	
It is obligatory that A	
A is provable in PA	
The program terminates with A	
A is a rational belief	

Interpretations of the box operator

$\Box A$	$\Diamond A$
It is necessary that A	It is possible that A
It will always be that A	It will be that A [non-branching]
It is known that A	
It is obligatory that A	
A is provable in PA	
The program terminates with A	
A is a rational belief	

Interpretations of the box operator

$\Box A$	$\Diamond A$
It is necessary that A	It is possible that A
It will always be that A	It will be that A [non-branching]
It is known that A	It is conceivable that A
It is obligatory that A	
A is provable in PA	
The program terminates with A	
A is a rational belief	

Interpretations of the box operator

$\Box A$	$\Diamond A$
It is necessary that A	It is possible that A
It will always be that A	It will be that A [non-branching]
It is known that A	It is conceivable that A
It is obligatory that A	It is permissable that A
A is provable in PA	
The program terminates with A	
A is a rational belief	

Interpretations of the box operator

$\Box A$	$\Diamond A$
It is necessary that A	It is possible that A
It will always be that A	It will be that A [non-branching]
It is known that A	It is conceivable that A
It is obligatory that A	It is permissable that A
A is provable in PA	A is consistent with PA
The program terminates with A	
A is a rational belief	

Interpretations of the box operator

$\Box A$	$\Diamond A$
It is necessary that A	It is possible that A
It will always be that A	It will be that A [non-branching]
It is known that A	It is conceivable that A
It is obligatory that A	It is permissable that A
A is provable in PA	A is consistent with PA
The program terminates with A	The program might output A
A is a rational belief	

Interpretations of the box operator

$\Box A$	$\Diamond A$
It is necessary that A	It is possible that A
It will always be that A	It will be that A [non-branching]
It is known that A	It is conceivable that A
It is obligatory that A	It is permissable that A
A is provable in PA	A is consistent with PA
The program terminates with A	The program might output A
A is a rational belief	A is plausible

Semantics

- A semantics offers an interpretation of the vocabulary of the language.
- Standardly, a semantics tells us under what conditions sentences are *true*.
- For example, truth tables tell us that $A \rightarrow B$ is true when A is false or B is true.

Semantics of Propositional Logic

- A *valuation* v is an assignment of truth values (T, F) to atomic propositions $p, q \ldots$.
- A valuation is like a line in the truth table (but only under the atomic proposition letters).
- In the case of propositional logic, we can now characterise the truth of any sentence under a valuation:

$$
\begin{array}{lll}
\models^v p & \text{iff} & v(p) = T \\
\models^v A \to B & \text{iff} & \not\models^v A \text{ or } \models^v B \\
\models^v A \wedge B & \text{iff} & \models^v A \text{ and } \models^v B \\
\models^v \neg A & \text{iff} & \not\models^v A
\end{array}
$$

- This is a generalisation of truth tables. We can read $\models^v A$ as meaning that A is true under valuation v.

Semantics of $S5$: structures and valuations

- $S5$ is a simple modal logic, it is often taken to be the logic of metaphysical necessity.
- A *structure* for $S5$, $\mathcal{S}_5$ is a just set of *worlds*.
- The worlds have no internal structure, in fact, they don't even have to be worlds, they could be dots, spacial points, beer glasses etc.
- A valuation v assigns a subset of $\mathcal{S}_5$ to each atomic proposition p.
- Think of v as specifying, for each *atomic* proposition, in which worlds it is true.

Semantics of $S5$

- Given a structure $\mathcal{S}_5$ and a valuation v, we can characterise the truth of any sentence at a world $s \in \mathcal{S}_5$:

$$
\begin{array}{lll}
\mathcal{S}_5 \models^v_s p & \text{iff} & s \in v(p) \\
\mathcal{S}_5 \models^v_s A \to B & \text{iff} & \mathcal{S}_5 \not\models^v_s A \text{ or } \mathcal{S}_5 \models^v_s B \\
\mathcal{S}_5 \models^v_s A \wedge B & \text{iff} & \mathcal{S}_5 \models^v_s A \text{ and } \mathcal{S}_5 \models^v_s B \\
\mathcal{S}_5 \models^v_s \neg A & \text{iff} & \mathcal{S}_5 \not\models^v_s A \\
\mathcal{S}_5 \models^v_s \Box A & \text{iff} & \mathcal{S}_5 \models^v_t A \text{ for all } t \in \mathcal{S}_5 \\
\mathcal{S}_5 \models^v_s \Diamond A & \text{iff} & \mathcal{S}_5 \models^v_t A \text{ for some } t \in \mathcal{S}_5
\end{array}
$$

- We can read $\mathcal{S}_5 \models^v_s A$ as meaning that A is true at world s under valuation v (in the structure $\mathcal{S}_5$).

Some definitions

- $\mathcal{S}_5 \models_s^v A$ means that A is true at the world s under valuation v.
- $\mathcal{S}_5 \models^v A$ means that A is true at *every* world, under v.
- $\mathcal{S}_5 \models A$ means that A is true at *every* world, under *any* v

  A is true in the structure $\mathcal{S}_5$

- $\models A$, or $S5 \models A$, means that A is true in every $S5$ structure, i.e. A is a logical truth of $S5$.

Some logical truths of $S5$

- $\Box A \rightarrow A$
- $\Box A \leftrightarrow \Box\Box A$
- $\Box A \rightarrow \neg\Box\neg A$
- $A \rightarrow \Diamond A$
- $\Box A \rightarrow \Box\Diamond A$
- $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- $\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$

Some non-truths of $S5$

- $A \rightarrow \Box A$
- $\Diamond A \rightarrow \Box A$
- $\Diamond A \rightarrow A$
- $\Box\Diamond A \rightarrow \Box A$
- $\Diamond(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$
- $\Diamond(A \wedge B) \leftrightarrow (\Diamond A \wedge \Diamond B)$