Modal Logic 2

Lecture Contents

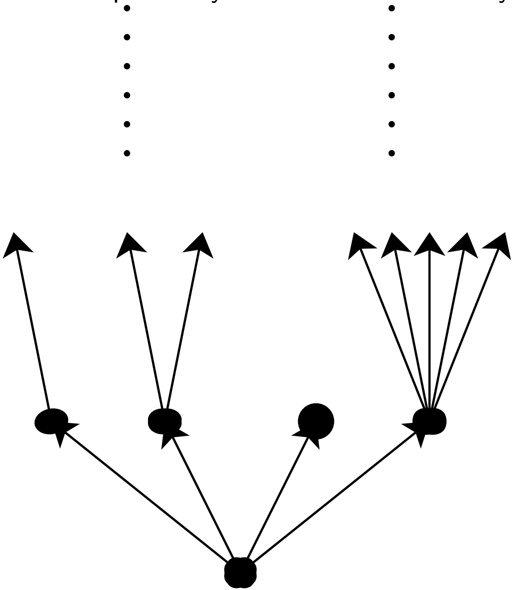
- The notion of a Kripke Frame and an accessability relation
- Structural properties of an accessability relation: reflexivity, transitivity, symmetry and more
- Truth in a Kripke frame
- The logics K, T, $S4$ and $S5$

Kripke frames

- To interpret more modal operators we need a more general notion of a structure.
- A *Kripke Frame* $\mathcal{F}$ consists of two parts (W, R):
  - W is a set of worlds
  - R is a binary relation on the worlds
- R is standardly called an *accessibility* relation.
- We get different modal logics by varying the conditions on the accessibility relation R.

Viewing the accessibility relation

The simplest way to view an accessibility relation is as a tree:



Kripke semantics

- A Kripke frame $\mathcal{F}$ is a pair (W, R), where W is a set of worlds and R is a relation on the worlds.
- A valuation v assigns a subset of W to each atomic proposition p.
- Truth at a world in a frame, under a valuation, is characterised as follows:

$$
\begin{array}{lll}
\mathcal{F} \vDash_w^v p & \text{iff} & w \in v(p) \\
\mathcal{F} \vDash_w^v A \rightarrow B & \text{iff} & \mathcal{F} \nvDash_w^v A \text{ or } \mathcal{F} \vDash_w^v B \\
\mathcal{F} \vDash_w^v A \wedge B & \text{iff} & \mathcal{F} \vDash_w^v A \text{ and } \mathcal{F} \vDash_w^v B \\
\mathcal{F} \vDash_w^v \neg A & \text{iff} & \mathcal{F} \nvDash_w^v A \\
\mathcal{F} \vDash_w^v \Box A & \text{iff} & \mathcal{F} \vDash_{w'}^v A \text{ for all } w' \in W \text{ s.t. } wRw' \\
\mathcal{F} \vDash_w^v \Diamond A & \text{iff} & \mathcal{F} \vDash_{w'}^v A \text{ for some } w' \in W \text{ s.t. } wRw'
\end{array}
$$

Semantics for $S5$

- A Kripke Frame for $S5$ is a pair (W, R):
  - W is a set of worlds
  - R is a *universal* binary relation on the worlds
- A binary relation R on W is universal when

$$wRw' \text{ for every } w, w' \in W$$

- Compare the two semantic clauses for $\Box A$:
  - $\mathcal{F} \models^v_w \Box A$ iff $\mathcal{F} \models^v_{w'} A$ for all $w' \in W$ s.t. wRw'
  - $\mathcal{S}_5 \models^v_w \Box A$ iff $\mathcal{S}_5 \models^v_{w'} A$ for all $w' \in \mathcal{S}_5$

Conditions on binary relations

Reflexive	$\forall w(wRw)$
Transitive	$\forall w_1 w_2 w_3\big((w_1 R w_2 \wedge w_2 R w_3) \rightarrow w_1 R w_3\big)$
Symmetric	$\forall w_1 w_2(w_1 R w_2 \rightarrow w_2 R w_1)$
Serial	$\forall w_1 \exists w_2(w_1 R w_2)$
Functional	$\forall w_1 \exists! w_2(w_1 R w_2)$
Weakly Dense	$\forall w_1 w_2\big(w_1 R w_2 \rightarrow \exists w_3(w_1 R w_3 \wedge w_3 R w_2)\big)$
Weakly Directed	$\forall w_1 w_2 w_3\big(w_1 R w_2 \wedge w_1 R w_3 \rightarrow \exists w(w_2 R w \wedge w_3 R w)\big)$

Semantics for K

- A Kripke Frame for K is a pair (W, R):
  - W is a set of worlds
  - R is any binary relation on the worlds in W.
- Some truths truths of K:
  - If A is true of K then $\Box A$ is true of K.
  - $\Box(A \to B) \to (\Box A \to \Box B)$
  - $\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$
  - $\Box(A \to B) \to (\Diamond A \to \Diamond B)$

Semantics for T

- A Kripke Frame for T is a pair (W, R):
  - W is a set of worlds
  - R is any *reflexive* binary relation on the worlds in W.
- Some truths truths of T:
  - If A is true of T then $\Box A$ is true of T.
  - $\Box(A \to B) \to (\Box A \to \Box B)$
  - $\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$
  - $\Box(A \to B) \to (\Diamond A \to \Diamond B)$
  - $A \to \Diamond A$
  - $\Box A \to A$

Semantics for $S4$

- A Kripke Frame for $S4$ is a pair (W, R):
  - W is a set of worlds
  - R is any *reflexive* and *transitive* binary relation on the worlds in W.
- Some truths truths of $S4$:
  - If A is true of $S4$ then $\Box A$ is true of $S4$.
  - $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
  - $\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$
  - $\Box(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$
  - $A \rightarrow \Diamond A$
  - $\Box A \rightarrow A$
  - $\Box A \rightarrow \Box\Box A$

Semantics for $S5$

- A Kripke Frame for $S5$ is a pair (W, R):
  - W is a set of worlds
  - R is any *reflexive*, *transitive* and *symmetric* binary relation (i.e. an *equivalence* relation) on the worlds in W.
- Some truths truths of $S5$:
  - If A is true of $S5$ then $\Box A$ is true of $S5$.
  - $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
  - $\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$
  - $\Box(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$
  - $A \rightarrow \Diamond A$
  - $\Box A \rightarrow A$
  - $\Box A \rightarrow \Box\Box A$
  - $A \rightarrow \Box\Diamond A$
- What is the difference between a universal relation and an equivalence relation?