Modal Logic 3

Lecture Contents

- The notion of an axiomatisation.
- The axioms for the *normal* modal logic K.
- The axioms $K, T, B, 4$ and 5.
- Soundness and correspondence of axioms to Kripke frames.

What is a formal logic

- A formal logic is a consequence relation $\vdash$ over expressions of a formal language.
- A formal language has a precise syntax and grammatical rules.
- It should be a decidable matter whether a given string of symbols is a well formed formulae of that language.
- Usually, a consequence relation $\vdash$ relates a set of wffs Γ and a single formula A:

  $$\Gamma \vdash A$$

  which intuitively means that A is a logical consequence of the formulae in Γ.

Logical consequence relations

- Consequence relations of interest are ones where we can specify under what conditions $\Gamma \vdash A$ holds, and verify that these conditions hold.
- In some cases, we can specify a way of verifying whether *or not* these conditions hold.
- For example we can take the language of propositional logic and define:

  $\Gamma \vdash A$ iff $\phi \rightarrow A$ is derivable using the system of Magnus' book

  where ϕ is some conjunction of formulae in Γ.

Axiomatisations

- Often it is convenient to define logics in terms of some simple inference rules and some axioms.
- Say that an *axiomatic derivation* that $\Gamma \vdash A$ is a finite sequence of wff that ends with A and is constructed according to the rules of the logic.
- Usually these rules are that derivations can be extended (or can begin):
  - with a member of Γ
  - with one of a decidable set of axioms.
  - according to one of a decidable set of decidable inference rules.
- A set is decidable if we can decide, in finite time, whether an object is a member of it or not.

Consequences vs Theorems

- The logical consequence relation $\Gamma \vdash A$ can be read 'A is deducible from Γ'
- A *theorem* of a logic is any formula that can be derived from no premises:

$$\vdash A$$

- Often, logics have implication $\rightarrow$, and we can describe a logic only by presenting its theorems.
- We can say that $\Gamma \vdash A$ holds when $(B_1 \wedge \ldots B_n) \rightarrow A$ is a theorem of the logic (where each $B_i \in \Gamma$).

Propositional logic

- For example, the theorems of propositional logic can be characterised axiomatically as follows:

$$A \to (B \to A)$$
$$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$$
$$A \to (\neg A \to B)$$
$$(A \to \neg A) \to \neg A$$
$$\neg\neg A \to A$$

$$\frac{A \quad A \to B}{B}$$

- An alternative would be to regard all tautologies as axioms (the tautologies are a decidable set)
- We can say that $\Gamma \vdash A$ in propositional logic when a suitable $(B_1 \wedge \ldots B_n) \to A$ is a tautology.

The Logic K

- K is the weakest modal logic than can be characterised by Kripke frames.
- We can axiomatise the theorems of K by:

$$(K)\quad \begin{array}{l} \mathit{AnyTautology} \\ \Box(A \to B) \to (\Box A \to \Box B) \\ \Diamond A \leftrightarrow \neg\Box\neg A \end{array} \qquad \frac{A \quad A \to B}{B}\,(MP) \quad \frac{A}{\Box A}\,(N)$$

- We can now fully characterise K by defining $\Gamma \vdash_K A$ to hold when there is a suitable theorem of K of the form $(B_1 \wedge \ldots B_n) \to A$ (for $B_i \in \Gamma$).

A warning about (MP) and (N)

- We have characterised K by axiomatising its *theorems*.
- The two inference rules take *theorems* as premises and output *theorems* as conclusions.
- Theorems will correspond to logical (semantic) truths.
- So (N) should be interpreted as:

  If A is a logical truth, then so is $\Box A$

- Also, (MP) as we are using it is not quite Modus Ponens.
- Nevertheless, $A, A \to B \vdash_K B$ does hold by (MP) and the fact that $\big(A \wedge (A \to B)\big) \to B$ is a tautology.

Normal Modal Logics

- A logic L' extends L if any consequence of L is also a consequence of L'.
- We have been characterising logics by their theorems, so a logic extends another by adding more theorems.
- Any logic that extends K is called a *normal modal logic*. K is the smallest, or weakest, modal logic.
- We get new modal logics by adding new axioms:

$$
\begin{array}{cl}
(T) & \Box A \rightarrow A \\
(B) & A \rightarrow \Box\Diamond A \\
(D) & \Box A \rightarrow \Diamond A \\
(Func) & \Diamond A \rightarrow \Box A \\
(Triv) & A \rightarrow \Box A \\
(4) & \Box A \rightarrow \Box\Box A \\
(5) & \Diamond A \rightarrow \Box\Diamond A
\end{array}
$$

Axioms and axiom schema

- Each of the 'axioms' is really an axiom scheme.
- For example the axiom scheme (T) corresponds to adding, as an axiom, every instance of the scheme

$$\Box A \rightarrow A$$

- Each axiom scheme has a *dual*, that is equivalent (as an axiom scheme). For example the dual of (T) is:

$$A \rightarrow \Diamond A$$

  The dual for (4) is:

$$\Diamond\Diamond A \rightarrow \Diamond A$$

- What is the dual for (D)?

Axiomatising famous normal modal logics

- Each logic extends K and so contains the axioms and rules of K as well as:

T	(T)	$\Box A \to A$
B	(B)	$A \to \Box\Diamond A$
$S4$	(T)	$\Box A \to A$
	(4)	$\Box A \to \Box\Box A$
$S5$	(T)	$\Box A \to A$
	(5)	$\Diamond A \to \Box\Diamond A$

$S5$ can also be axiomatised as $KT4B$.