Modal Logic 4
Lecture Contents

- The notions of soundness and correspondence
- The notion of completeness
- Sketch of proofs of soundness, correspondence and completeness
Characterising logics

- We have characterised the logics $K$ semantically and syntactically.
- We said that the logical truths of $K$ are all those formulae that hold in any Kripke frame $\mathcal{F}$.
- We also said that the theorems of $K$ are all those derivable from a particular axiom system.
- But do the two characterisations match?
- Are the theorems exactly the logical truths?
Soundness and Completeness

- Given our semantics for $K$, we can look at its axioms and ask
  1. Is every theorem of $K$ a logical truth?
  2. Is every logical truth of $K$ a theorem?
- If (1) holds then the axioms of $K$ are *sound*.
- If (2) holds then the axioms of $K$ are *complete*. 
Correspondence

- We have also looked briefly at a related question.
- Given a formula $A$, we can ask what kind of Kripke frame will satisfy $A$ under any valuation $v$.
- More specifically, what kind of frame $\mathcal{F}$ is such that $\mathcal{F} \models A$?
- Remember that $\mathcal{F} \models A$ means that $\mathcal{F} \models^v_w A$ for every $w$ given any $v$.
- So equivalently, what kind of frame $\mathcal{F}$ is such that $\mathcal{F} \models^v_w A$ in all $w$ under any $v$. 
An example

- Remember axioms for the logic $T$. In particular the axiom:
  \[ \square A \rightarrow A \]

- We can then ask:
  - Are all the axioms of $T$ satisfied by reflexive Kripke frames (soundness)
  - Is any Kripke Frame that satisfies $\square A \rightarrow A$ reflexive? (correspondence)
  - Are all the formulae satisfied by all reflexive Kripke frames derivable from the axioms of $T$?
Proving Soundness (of $T$)

We want to show that if $\vdash_T A$ then $\mathcal{F} \models^\nu_w A$ for every $\nu, w, \mathcal{F}$ where $R \in \mathcal{F}$ is reflexive.

We must check that:

1. If $A$ is any instance of an axiom ($K$) or ($T$) then $\mathcal{F} \models A$ in any reflexive $\mathcal{F}$.
2. If the premises of an inference rule ($N$) or ($MP$) are satisfied by any reflexive frame $\mathcal{F}$ then so are the conclusions.

Then we will have shown that $\vdash_T A$ implies $\mathcal{F} \models A$ for any reflexive frame.
Something stronger

- Actually we can prove something stronger:
- If $\Gamma \vdash_T A$ then for every $v, w, \mathcal{F}$ where $R \in \mathcal{F}$ is reflexive, if $\mathcal{F} \models^v_w B$ for all $B \in \Gamma$, then $\mathcal{F} \models^v_w A$. 
Proving correspondence (of $T$)

- We want to show that if $\mathcal{F} \models \Box p \rightarrow p$, then $\mathcal{F}$ is reflexive.
  - Take any $w \in \mathcal{F}$, and consider $\nu$ such that
    $\nu(p) = \{w' | wRw'\}$
  - Then $\mathcal{F} \vdash^\nu \Box p$
  - But since $\mathcal{F} \vdash^\nu \Box p \rightarrow p$ for all $\nu$, we must have $\mathcal{F} \vdash^w \Box p$.
  - So $\mathcal{F} \vdash^\nu \Box p$ and so $w \in \{w' | wRw'\}$, so $wRw'$.

- Not all correspondence results are quite so simple.
Proving completeness for \( T \)

- We want to show that if \( \mathcal{F} \models^v_w A \) for every \( v, w, \mathcal{F} \) where \( R \in \mathcal{F} \) is reflexive, then \( \vdash_T A \).
- The standard proof assumes \( \not\vdash_T A \) and shows that some \( \mathcal{F} \not\models A \) for some reflexive \( \mathcal{F} \).
- Say that a set of formulae \( \Gamma \) is \textit{consistent} when no contradiction can be derived from it.
- So \( \Gamma \) is consistent in \( T \) if \( \Gamma \not\vdash_T A \land \neg A \).
- Say that \( \Gamma \) is \textit{complete} when, for every formula \( A \), either \( A \in \Gamma \) or \( \neg A \in \Gamma \).
- Notice that if \( \Gamma \) is complete and consistent then if \( \Gamma \vdash_T A \) then \( A \in \Gamma \).
Canonical frames

- First we show that if $\not\models_T A$ then there is a complete consistent set that contains $\neg A$.
- Now we build a Kripke frame out of all the complete consistent sets.
- We need to specify $\mathcal{F}_T = (W_T, R_T)$
  - We set $W_T$ to be the set of all complete consistent sets.
  - We set $wR_T w'$ just in case $A \in w'$ whenever $\Box A \in w$ (for all formulae $A$).
- Notice that since $\Box A \rightarrow A$ is a theorem of $T$, it follows that $wR_T w$ for all $w$. 
The canonical valuation

- Now we set \( v(p) = \{ w \in W_T \mid p \in w \} \)
- We then verify that \( F_T \models^v_w A \) iff \( A \in w \)
- And now, if \( \not\models_T A \), then there is a complete consistent set \( w \) which contains \( \neg A \).
- \( w \) is one of the worlds in the reflexive frame \( F_T \).
- But, since \( A \notin w \) it follows that \( F_T \not\models^v_w A \) under the valuation \( v \) described above.
- So if \( \not\models_T A \) then \( F \not\models^v_w A \) for some \( v, w \) in some reflexive \( F \).
Something stronger

- Actually we can prove something stronger:
  - If for every $v, w, \mathcal{F}$ where $R \in \mathcal{F}$ is reflexive, $\mathcal{F} \models^v_w B$ for all $B \in \Gamma$ implies $\mathcal{F} \models^v_w A$, then $\Gamma \vdash_T A$. 
It is not just for $T$

- We can show that many axiomatisations are sound, complete and correspond to Kripke frames with certain conditions on the accessibility relation $R$.

  $K$ Kripke frames
  $T$ reflexive Kripke frames $\quad \Box A \rightarrow A$
  $B$ symmetric Kripke frames $\quad A \rightarrow \Box \Diamond A$
  4 transitive Kripke frames $\quad \Box A \rightarrow \Box \Box A$
  5 Euclidean Kripke frames $\quad \Diamond A \rightarrow \Box \Diamond A$
  $D$ Serial Kripke frames $\quad \Box A \rightarrow \Diamond A$

- We can also show that combining axioms combines the properties. So

  $S4 = KT4$

  and

  $S5 = KT4B = KT5$

- In fact, we can show that all these logics are complete for finite frames, implying they are all decidable.