Modal Logic 4

Lecture Contents

- The notions of soundness and correspondence
- The notion of completeness
- Sketch of proofs of soundness, correspondence and completeness

Characterising logics

- We have characterised the logics K semantically and syntactically
- We said that the logical truths of K are all those formulae that hold in any Kripke frame $\mathcal{F}$.
- We also said that the theorems of K are all those derivable from a particular axiom system.
- But do the two characterisations match?
- Are the theorems *exactly* the logical truths?

Soundness and Completeness

- Given our semantics for K, we can look at its axioms and ask
  1. Is every theorem of K a logical truth?
  2. Is every logical truth of K a theorem?
- If (1) holds then the axioms of K are *sound*.
- If (2) holds then the axioms of K are *complete*.

Correspondence

- We have also looked briefly at a related question.
- Given a formula A, we can ask what kind of Kripke frame will satisfy A under any valuation v.
- More specifically, what kind of frame $\mathcal{F}$ is such that $\mathcal{F} \vDash A$
- Remember that $\mathcal{F} \vDash A$ means that $\mathcal{F} \vDash_w^v A$ for every w given any v.
- So equivalently, what kind of frame $\mathcal{F}$ is such that $\mathcal{F} \vDash_w^v A$ in all w under any v.

An example

- Remember axioms for the logic T. In particular the axiom:

$$\Box A \rightarrow A$$

- We can then ask:
  - Are all the axioms of T satisfied by reflexive Kripke frames (soundness)
  - Is any Kripke Frame that satisfies $\Box A \rightarrow A$ reflexive? (correspondence)
  - Are all the formulae satisfied by all reflexive Kripke frames derivable from the axioms of T?

Proving Soundness (of T)

- We want to show that if $\vdash_T A$ then $\mathcal{F} \models^v_w A$ for every $v, w, \mathcal{F}$ where $R \in \mathcal{F}$ is reflexive.
- We must check that:
  1. If A is any instance of an axiom (K) or (T) then $\mathcal{F} \models A$ in any reflexive $\mathcal{F}$.
  2. If the premises of an inference rule (N) or (MP) are satisfied by any reflexive frame $\mathcal{F}$ then so are the conclusions.
- Then we will have shown that $\vdash_T A$ implies $\mathcal{F} \models A$ for any reflexive frame.

Something stronger

- Actually we can prove something stronger:
- If $\Gamma \vdash_T A$ then for every $v, w, \mathcal{F}$ where $R \in \mathcal{F}$ is reflexive, if $\mathcal{F} \vDash_w^v B$ for all $B \in \Gamma$, then $\mathcal{F} \vDash_w^v A$.

Proving correspondence (of T)

- We want to show that if $\mathcal{F} \models \Box p \to p$, then $\mathcal{F}$ is reflexive.
  - Take any $w \in \mathcal{F}$, and consider v such that $v(p) = \{w' \mid wRw'\}$
  - Then $\mathcal{F} \vdash^v_w \Box p$
  - But since $\mathcal{F} \vdash^v \Box p \to p$ for all v, we must have $\mathcal{F} \vdash^v_w \Box p$.
  - So $\mathcal{F} \vdash^v_w p$ and so $w \in \{w' \mid wRw'\}$, so wRw'.
- Not all correspondence results are quite so simple.

Proving completeness for T

- We want to show that if $\mathcal{F} \vDash_w^v A$ for every $v, w, \mathcal{F}$ where $R \in \mathcal{F}$ is reflexive, then $\vdash_T A$.
- The standard proof assumes $\nvdash_T A$ and shows that some $\mathcal{F} \nvDash A$ for some reflexive $\mathcal{F}$.
- Say that a set of formulae Γ is *consistent* when no contradiction can be derived from it.
- So Γ is consistent in T if $\Gamma \nvdash_T A \wedge \neg A$.
- Say that Γ is *complete* when, for every formula A, either $A \in \Gamma$ or $\neg A \in \Gamma$.
- Notice that if Γ is complete and consistent then if $\Gamma \vdash_T A$ then $A \in \Gamma$.

Canonical frames

- First we show that if $\nvdash_T A$ then there is a complete consistent set that contains $\neg A$.
- Now we build a Kripke frame out of all the complete consistent sets.
- We need to specify $\mathcal{F}_T = (W_T, R_T)$
  - We set W_T to be the set of all complete consistent sets.
  - We set $wR_T w'$ just in case $A \in w'$ whenever $\Box A \in w$ (for all formulae A).
- Notice that since $\Box A \to A$ is a theorem of T, it follows that $wR_T w$ for all w.

The canonical valuation

- Now we set $v(p) = \{w \in W_T \mid p \in w\}$
- We then verify that $\mathcal{F}_T \vDash^v_w A$ iff $A \in w$
- And now, if $\nvdash_T A$, then there is a complete consistent set w which contains $\neg A$.
- w is one of the worlds in the reflexive frame $\mathcal{F}_T$.
- But, since $A \notin w$ it follows that $\mathcal{F}_T \nvDash^v_w A$ under the valuation v described above.
- So if $\nvdash_T A$ then $\mathcal{F} \nvDash^v_w A$ for some v, w in some reflexive $\mathcal{F}$.

Something stronger

- Actually we can prove something stronger:
- If for every $v, w, \mathcal{F}$ where $R \in \mathcal{F}$ is reflexive, $\mathcal{F} \vDash_w^v B$ for all $B \in \Gamma$ implies $\mathcal{F} \vDash_w^v A$, then $\Gamma \vdash_T A$.

It is not just for T

- We can show that many axiomatisations are sound, complete and correspond to Kripke frames with certain conditions on the accessibility relation R.

K	Kripke frames	
T	reflexive Kripke frames	$\Box A \to A$
B	symmetric Kripke frames	$A \to \Box\Diamond A$
4	transitive Kripke frames	$\Box A \to \Box\Box A$
5	Euclidean Kripke frames	$\Diamond A \to \Box\Diamond A$
D	Serial Kripke frames	$\Box A \to \Diamond A$

- We can also show that combining axioms combines the properties. So

$$S4 = KT4$$

and

$$S5 = KT4B = KT5$$

- In fact, we can show that all these logics are complete for *finite* frames, implying they are all decidable.