Theories
Lecture four

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1 David Hilbert

- For Hilbert, theories are set of *partially-interpreted sentences* closed under *syntactic deduction*, $\vdash$

- His axioms are incapable of truth or falsity, since they have meaningless parts

- His primary notion of consistency is also *syntactic*:

  - a set of partially-interpreted sentences Γ is Hilbert-consistent iff there is no ϕ such that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$

- A Hilbert-style consistency proof is always a *relative* consistency proof

  - 'x is a point' is assigned the set of pairs $\langle x, y \rangle$ of real numbers
  - 'x is a line' is assigned the set of ratios $[u : v : w]$ of real numbers
  - 'x lies on y' is assigned the set of pairs $\langle \langle x, y \rangle, [u : v : w] \rangle$ such that $ux + vy + w = 0$
  - 1 For any two points there exists at most one line on which those points lie
  - I$'$ For any pair of pairs of real numbers $\langle \langle a, b \rangle, \langle c, d \rangle \rangle$, there is at most one ratio of real numbers $[e : f : g]$ such that $ae + bf + g = 0$ and $ce + df + g = 0$

- There is a related notion of *property-consistency*

- Hilbert's syntactic consistency proofs entail property-consistency and, if the deductive system is *complete*, property-consistency entails syntactic consistency

- He believes that his axioms *implicitly define* the nonlogical primitives

- This whole approach is close to modern orthodoxy

2 Gottlob Frege

- Frege's semantics involved a threefold division between *language* (the strings of symbols), *reference* (the world), and *sense* (roughly, how a linguistic expression presents its reference)

- At the level of language, *first-level predicates* are formed by deleting a name form a sentence and replacing it with a variable. *Second-level predicates* are formed by deleting a first-level predicate from a sentence and replacing it with a variable

- At the level of reference, first-level predicates pick out *first-level concepts*, and second-level predicates pick our *second-level concepts*

- At the level of sense, the sense of a sentence is a *proposition* (in Frege's terminology, a *thought*)

- His theories are sets of *propositions* closed under Frege-consequence, $\models_F$, which is at the level of sense

- Similarly, Frege-consistency is at the level of sense:

  - a set of propositions Γ is Frege-consistent iff there is no proposition ϕ such that $\Gamma \models_F \phi$ and $\Gamma \models_F \neg\phi$

- Many Frege-consequences are not syntactic consequences, so syntactic consistency is not a reliable guide to Frege-consistency

- To demonstrate Frege-consistency, we must present a model

3 The disagreement

- For Hilbert, *consistency implies truth*

- For Frege, *truth implies consistency*

- For Frege, relative consistency proofs are illegitmate, since they involve changing the subject

- By Frege's lights, Hilbert does not provide implicit definitions of his nonlogical primitives, but of *second-level concepts*

- But Hilbert's approach is more fruitful, and has become standard

- The disagreement comes down to fundamental differences in the nature of logic