Theories
Frege and Hilbert

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Previously on *Theories*

- Last week, we considered the notions of *consistency* and *independence* in the context of geometric theories.
- We saw that it is difficult to show that a theory is absolutely consistent, and that frequently we must give proofs of *relative* consistency.
- We sketched a proof that hyperbolic geometry is consistent relative to Euclidean geometry.
- We saw that it is a corollary of this result that Euclid’s controversial Axiom 5 is *independent* of the other axioms: there is no proof of Axiom 5 from the other axioms.
- Generally, independence is a desirable property of axioms: it ensures that our axiom sets are as lean as possible.
Talk outline

1. David Hilbert
2. Gottlob Frege
3. The disagreement
4. Who wins?
5. Conclusion
Gottlob Frege and David Hilbert first met in 1895
Between 1st October 1895 and 7th November 1903, they exchanged a number of letters
It began with Frege trying to justify the use of formalisation in mathematics, logic and science. Hilbert, of course, agreed
But in his letter dated 27th December 1899, Frege began to criticise Hilbert’s masterpiece *Die Grundlagen der Geometrie*, in particular for its use of *relative consistency* and *implicit definition*
Then Hilbert defended himself, and Frege tried to reformulate his criticisms
After Frege’s reformulation, Hilbert stopped responding
This lecture will largely follow Patricia Blanchette’s excellent ‘Frege and Hilbert on Consistency’, 1996, *Journal of Philosophy*
Hilbert is the easier for us to understand today, because his conception of theories has become orthodoxy.

For Hilbert, theories are sets of *partially interpreted sentences* that are closed under *syntactic* deduction.

Whether a formula $\phi$ is entailed by $\psi$ is not, then, a matter of what they mean.

As such, his axioms are not capable of truth or falsity: to be true or false, a sentence must have a determinate meaning, but partially-interpreted sentences have uninterpreted parts, and so are meaningless.
Unsurprisingly, Hilbert’s notion of consistency was also primarily syntactic:

- a set of partially interpreted sentences $\Gamma$ is Hilbert-consistent iff there is no $\phi$ such that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg \phi$

A consistency proof for Hilbert is always a *relative* consistency proof.
Relative consistency 1

- Hilbert, as we saw last week, axiomatised Euclidean and hyperbolic geometry, and proved the latter consistent relative to the former.
- He also showed that both were consistent relative to the theory of real numbers.
- A real number is a value that represents a quantity along a continuous line. The real numbers are the rational numbers (those that can be expresses as a fraction) and all the irrational numbers (those that cannot be expressed as a fraction, e.g. \( \pi \)).
- Recall that a proof that \( \Theta_2 \) is consistent relative to \( \Theta_1 \) comes in two stages: (i) interpret the nonlogical primitives of \( \Theta_2 \) in the language of \( \Theta_1 \); (ii) show that, so interpreted, the axioms of \( \Theta_2 \) are theorems of \( \Theta_1 \).
To show that hyperbolic geometry is consistent relative to the theory of real numbers, Hilbert first interpreted the axioms as follows:

- ‘x is a point’ is assigned the set of pairs \( \langle x, y \rangle \) of real numbers
- ‘x is a line’ is assigned the set of ratios \([u : v : w]\) of real numbers
- ‘x lies on y’ is assigned the set of pairs \(\langle \langle x, y \rangle, [u : v : w] \rangle\) such that \(ux + vy + w = 0\)

The next stage is to show that, so interpreted, the axioms of hyperbolic geometry are theorems of the theory of real numbers
Relative consistency 3

- E.g. consider the axiom:
  1. For any two points there exists at most one line on which those points lie

- Hilbert interpreted this as:
  1'. For any pair of pairs of real numbers \(\langle\langle a, b\rangle, \langle c, d\rangle\rangle\), there is at most one ratio of real numbers \([e : f : g]\) such that:

\[
ae + bf + g = 0 \quad \text{and} \quad ce + df + g = 0
\]

- He then proves that 1' is a theorem of the theory of real numbers

- Therefore, hyperbolic geometry is consistent relative to the theory of real numbers
The idea of a relative consistency proof fits naturally with Hilbert’s understanding of consistency as syntactic.

Syntactic consistency pays no attention to the meanings of the nonlogical primitives.

So, if $\Theta_2$ is consistent on an interpretation of its nonlogical primitives, then we know that its consistency is entirely due to the logical vocabulary.

And in this case, then however we interpret the nonlogical primitives, we can be confident that the theory will be consistent, since the logical vocabulary has remained unchanged.
Hilbert also thinks there is a related notion of property-consistency.

Consider the following two axioms sets:

\[\Sigma_1 \{ \forall x \exists y Rxy; \neg \exists y \forall x Rxy \}\]
\[\Sigma_2 \{ \neg \forall x \exists y Rxy; \exists y \forall x Rxy \}\]

There will of course be many interpretations of the predicate \(Rxy\) that satisfies \(\Sigma_1\), e.g. let ‘\(Rxy\)’ mean that \(x = y\).

But there are no interpretations that satisfy \(\Sigma_2\).

Hilbert thought that satisfiable sets define complex preoperties. If there are \(n\) nonlogical primitives in a set of sentences, then that set defines an \(n\)-place complex property.

These axioms define, in his words, a ‘scaffolding of concepts’.
What is the relation between syntactic consistency and property-consistency?

Hilbert points out that syntactic consistency proofs establish property-consistency.

And he notes that, if the deductive system is complete, then property-consistency will entail syntactic consistency.
Normal, uncontroversial definitions are *explicit*.

An explicit definition of a new symbol is an abbreviation of another, already understood symbol.

E.g. we can explicitly define the predicate ‘x is even’ as follows: ‘x is even’ has the same meaning as ‘there is some integer $n$ such that $x = 2n$’.

Crucially, when we explicitly define a symbol, we *mention* rather than *use* it.
Implicit definition

- We *implicitly* define a new symbol, on the other hand, when we *use* the expression.

- E.g. some London police officer in the 1890s may implicitly define ‘Jack the Ripper’ to mean whatever it has to mean in order for ‘Jack the Ripper committed the Whitechapel murders’ to be true.

- Likewise, we can *implicitly* define a ‘point’ to mean whatever it has to mean in order for the axioms of Euclidean geometry to be true.

- Hilbert believed that the axioms of his mathematical theories *implicitly* define the primitive terms.
Hilbert in summary

- Theories are sets of *partially interpreted sentences* closed under *syntactic* deduction
- Consistency is *syntactic* consistency and gets proved by *relative* consistency proofs
- The axioms of mathematical theories *implicitly define* the nonlogical primitives
- There is a related notion of *property-consistency*: satisfiable sets of axioms define complex properties
- This whole approach is very close to today’s orthodoxy
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First, some quick Frege revision

Frege’s semantics famously involves three levels: language, sense and reference

The level of *language* features sentences which are just strings of symbols built according to some grammatical rules

*Sentences* can be broken down into names and predicates

*Names* are symbols like ‘Frege’ and ‘Hilbert’
First-level predicates are formed by deleting a name from a sentence and replacing it with a variable, e.g. from ‘Frege is a philosopher’, we can delete the name ‘Frege’ to form the first-level predicate ‘\(x\) is a philosopher’

Second-level predicates are formed by deleting a first-level predicate from a sentence and replacing it with a variable, e.g. from the sentence ‘\(\forall x (x\) is a logician)’, we can form the second-level predicate ‘\(\forall x Xx\)’

The notion of a second-level predicate will be important in Frege’s criticisms of Hilbert
Reference

- The level of *reference* is the level of the world
- Names refer to *objects*, e.g. ‘Frege’ refers to Frege
- First-level predicates refer to *first-level concepts*
- Second-level predicates refer to *second-level concepts*
- Be careful: talk of ‘concepts’ may *sound* like we’re at the level of sense but, for Frege, they are in the world. In particular, they are functions from objects to truth-values
- For completion, sentences also have reference for Frege. They refer to truth-values
Finally, there is the level of *sense*

It is hard to specify what senses are (it may be impossible)

But, roughly, the *sense* of an expression is the way that the expression presents its referent

The sense of Frege is then the way that ‘Frege’ presents Frege. This distinguishes ‘Frege’ from ‘the author of *Die Grundlagen der Arithmetik*’, both of which refer to Frege, but which present him in different ways

Names and predicates have sense, but so do whole sentences. The sense of a sentence is a *thought*, which we will call *propositions*, following modern usage
For Frege, theories are sets of *propositions* (level of *sense*) that are deductively closed.

But what is the relevant sense of deduction? It can’t be syntactic because that is at the level of language. But it also cannot be Hilbert’s property-consistency, since that was at the level of reference.

Instead, it must be a notion of deduction at the level of *sense*. Let’s call this *Frege-consequence*, $\models_F$. 
Unsurprisingly, then, Frege also thought of consistency at the level of sense:

- a set of sentences \( \Gamma \) is Frege-consistent iff there is no \( \phi \) such that \( \Gamma \models F \phi \) and \( \Gamma \models F \neg \phi \)

Consider the set:

\[
F \{ \text{Frege is a bachelor; Frege is not a man} \}
\]

For Frege, the proposition that Frege is a man is a Frege-consequence of the proposition that Frege is a bachelor. So, for Frege, \( F \) is a Frege-inconsistent set: it entails ‘Frege is a man’ and ‘Frege is not a man’

But of course in our usual syntactic practice, ‘Frege is a bachelor’ does not entail ‘Frege is a man’, so the set is not inconsistent.
Of course, Frege was also interested in sentences.
A good deductive system is, for Frege, one such that for any sentences $\phi$ and $\psi$, $\psi$ is derivable from $\phi$ iff the proposition expressed by $\psi$ is a Frege-consequence of the proposition expressed by $\phi$.

To return to the previous example, the sentence ‘Frege is a man’ is not a syntactic consequence of ‘Frege is a bachelor’, even though the proposition expressed by the former is a Frege-consequence of the proposition expressed by the latter.

Therefore, syntactic derivability is not, for Frege, a reliable guide to Frege-consequence.
Fregean consistency

• For this reason, Frege was not happy with proofs of syntactic consistency: they do not guarantee Frege-consistency, which is what we are really interested in.

• How, then, would Frege show that a theory was consistent?

• He insists that the only way to show that some propositions are consistent is to present something (a model) that satisfies them.

• A model, in this context, is an object of which all the axioms are true.
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Hilbert on consistency

- In his letter of 29th December 1899, Hilbert wrote that: *If the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence.*

- In a slogan: *consistency implies truth*
Frege on consistency and truth

- Of course, Frege rejects this idea.
- For him, theories consist of propositions, which have a fixed meaning and must be true or false.
- Hilbert-style consistency is an unreliable guide, as it does not guarantee Frege-consistency.
- For Frege, we show that a theory is consistent by presenting an object (model) that satisfies the axioms.
- If such a model can be given, then they are true and so consistent.
- In a slogan: *truth implies consistency*.
Since, for Frege, axioms must be meaningful and true, he rejects the idea that Hilbert’s partially-interpreted sentences can be axioms.

Further, since theories have a fixed subject matter, when we reinterpret axioms for a proof of relative consistency, we have changed the subject.

We cannot possibly draw conclusions about points and lines by demonstrating results about the theory of real numbers. They are completely separate theories, made up of completely separate propositions.
In his letter of 27th December 1899, Frege essentially offers a dilemma for Hilbert’s views on implicit definition.

Either all of the expressions in Hilbert’s axioms have meanings, or some of them do not.

If they all have meanings, then those axioms cannot possibly serve to define the meaning of any nonlogical expression: they all have meanings already!

If some of the expressions don’t have meanings, then the things that Hilbert calls ‘axioms’ aren’t really axioms: axioms must be true, so they cannot have meaningless parts.
But things get worse. Frege also contends that Hilbert’s axioms cannot possibly implicitly define the nonlogical primitives, since more than one nonlogical primitive features in each axiom:

In a letter of 6th January 1900, he wrote:

*Your system of definitions is like a system of equations with several unknowns, where there remains doubt whether the equations are soluble and, especially, whether the quantities are uniquely determined*
To illustrate the point, consider equations such as ‘\(x + y + z = 85\)’ and ‘\(x \times y = 49\)’. Here, it could be that \(x=7, \ y=7\) and \(z=71\), or that \(x=49, \ y=1\) and \(z=35\).

Recall that, in implicit definition, we define a primitive as meaning whatever it has to for certain sentences to be true. We could not say that these equations implicitly define ‘\(x\)’, since they do not specify any one meaning.

Hilbert’s axioms, Frege thinks, are just like these equations: they contain multiple variables and so cannot possibly be said to settle the meanings of the primitives.
Frege on implicit definition 4

- Hilbert has not, therefore, pinned down a meaning for nonlogical predicates such as ‘x is a point’, ‘x is a line’, etc.
- Rather, Frege claims that Hilbert has defined *second-level concepts*
- In effect, the nonlogical primitives in Hilbert’s axioms are *variables*, which can be interpreted in many ways, e.g. in geometric terms, or in terms of real numbers
- And these are variables that can be instantiated with *first-level predicates*
- But this is just the definition of a second-level predicate: a sentence from which first-level predicates have been deleted and replaced with variables
- Hilbert’s axioms, then, are complex second-level predicates, which express second-level concepts
Frege on implicit definition 5

- So Hilbert has not defined the first-level predicates that he intended to, but rather second-level predicates of a complex sort.
- Because Frege realised that Hilbert-style axioms are second-level predicates, it is natural for him to think that those predicates are mutually consistent by presenting a model of them: an object that all of those predicates are true of.
- Hilbert showed the second-level concept defined by the axioms of hyperbolic geometry, $H$, is consistent by finding some first-level concepts that fall under them (namely, concepts of real numbers).
- And this just supports Frege’s claim that we have not implicitly defined the primitives: we have found some first-level concepts that satisfy some second-level concept.
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The major points on which Frege and Hilbert disagree, therefore, are on the relationship between *consistency* and *truth*, and on Hilbert's claims about *implicit definition*

These disagreements flow from fundamental disagreements about the nature of axioms and consequence.

Indeed, when the two writers mean different things by the fundamental notions in the debate, it is unsurprising that they reached different conclusions, and struggled to understand each other.
There is a good sense in which Hilbert won the debate. This is because his methods are certainly more fruitful than Frege's. So fruitful, in fact, that they have become the standard. And it is because they are standard that we, as modern readers, may struggle even to understand what Frege meant.
Frege wins

- But Frege made some valuable points
- In particular, he seems correct that Hilbert’s axioms cannot implicitly define their nonlogical primitives
- Hilbert failed to appreciate the distinction between first- and second-level predicates and concepts, and so to realise that his axioms are really second-level predicates
Ultimately, however, since Frege and Hilbert meant different things by the central notions in the debate, the disagreement is deeper.

Concepts like *logical consequence* and *consistency* are the core notions of logic, so Frege and Hilbert represent radically different ways of thinking about the subject.

For Hilbert, these concepts hold primarily of sets of partially-interpreted sentences at the level of language.

For Frege-picture, they hold primarily of sets of propositions at the level of sense.

Whose conception of logic should we prefer? Can we use both? Questions for another time.
Talk outline

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We have seen that Frege and Hilbert disagreed fundamentally about the nature of mathematical theories.

The major points of disagreement are (i) the relation between consistency and truth, and (ii) whether axioms can implicitly define their primitives.

On the former, Hilbert’s methods seem more useful, and have become standard.

Frege was correct, however, on the second point.

Ultimately, though, who we deem the winner will depend on our conception of the nature of logic.