Consider the following modal language $L$:

$$A ::= p \mid \neg A \mid B \rightarrow C \mid B \land C \mid B \lor C \mid \Box A \mid \Diamond A$$

1. What does the symbol $p$ represent in the description of $L$ above?

2. A Kripke frame $(W, R)$ has two components. What are $W$ and $R$?

3. A Kripke model for $L$ consists of a Kripke frame and a valuation $v$. What is $v$?

4. If $\mathcal{F}$ is a Kripke frame and $v$ is a valuation, state the meaning of:
   (a) $\mathcal{F} \vDash^v_w A$
   (b) $\mathcal{F} \vDash^v A$
   (c) $\mathcal{F} \vDash A$

5. (a) Characterise the conditions under which $\mathcal{F} \vDash^v_w p$ holds for any atomic formula $p$ of $L$.
   (b) Characterise the conditions under which $\mathcal{F} \vDash^v_w A \land B$ and $\mathcal{F} \vDash^v_w A \rightarrow B$
      hold for any formulae $A$ and $B$.
   (c) Characterise the conditions under which $\mathcal{F} \vDash^v_w \Box A$ and $\mathcal{F} \vDash^v_w \Diamond A$
      hold for any formula $A$.

6. (a) Describe a Kripke model in which $\Diamond p \rightarrow \Box \Diamond p$ is false at some world.
   (b) Describe a Kripke model in which $\Diamond p \rightarrow \Diamond p$ is false at some world.
   (c) Describe a Kripke model in which $\Box (A \lor B) \rightarrow (\Box A \lor \Box B)$ is false at some world.
   (d) Argue that $\mathcal{F} \vDash \Diamond (A \lor B) \leftrightarrow (\Diamond A \lor \Diamond B)$ for any Kripke frame $\mathcal{F}$.
   (e) Argue that, if $\mathcal{F}$ is transitive, then $\mathcal{F} \vDash \Diamond \Diamond p \rightarrow \Diamond p$.
   (f) Describe a condition on the accessibility relation of a Kripke frame $\mathcal{F}$ under which $\mathcal{F} \vDash \Diamond p \rightarrow \Diamond p$.

7. Describe classes of Kripke frames that can be used to characterise the modal logics $T$, $S4$, and $S5$.

8. Suppose that $w$ is a world in a Kripke frame $\mathcal{F}$ and that, for any formula $A$, $\mathcal{F} \vDash A \rightarrow \Box \Diamond A$. Let $v$ be any valuation such that $v(p) = \{w\}$, argue that if $wRw'$ then $w'Rw$. Now conclude that $\mathcal{F}$ is symmetric.

9. (a) Let $\mathcal{F}$ be a frame such that, for any $w_1, w_2$, if $w_1 R w_2$ then there is no $w$ such that $w_2 R w$. Argue that $\mathcal{F} \vDash \Box \Box (A \land \neg A)$.
   (b) Describe a Kripke model in which $\mathcal{F} \vDash (A \land \neg A)$. 
The theorems of the model logic $K$ are axiomatised by the schemas:

\[
\begin{align*}
\text{(K)} & \quad \lozenge(A \rightarrow B) \rightarrow (\lozenge A \rightarrow \lozenge B) \\
\lozenge A \leftrightarrow \neg \lozenge \neg A & \\
A \rightarrow B & \\
\frac{A}{B} & \quad \text{(MP)} \\
\frac{A}{\lozenge A} & \quad \text{(N)}
\end{align*}
\]

1. What is an axiom schema?

2. Suppose $B$ is a tautological consequence of $A_1 \ldots A_n$, and suppose each $A_i$ is a theorem of $K$. Show that $B$ is also a theorem of $K$.

3. Explain why the rule (N) does not imply that $p \rightarrow \lozenge p$ is a theorem of $K$.

4. Write down additional axioms for the logics $T, S4$ and $S5$.

5. Suppose $A \rightarrow B$ is a theorem of $K$ (for some $A$ and some $B$), show that so is $\lozenge A \rightarrow \lozenge B$ and $\lozenge A \rightarrow \lozenge B$.

6. (a) Explain why $p \land q \rightarrow p$ is a theorem of $K$.

(b) Show that $\lozenge(p \land q) \rightarrow \lozenge p$ is a theorem of $K$.

(c) Conclude that $\lozenge(p \land q) \rightarrow (\lozenge p \land \lozenge q)$ is a theorem of $K$.

7. Show that $(\lozenge p \land \lozenge q) \rightarrow \lozenge(p \land q)$ is a theorem of $K$.

8. Show that $\lozenge(p \lor q) \leftrightarrow (\lozenge p \lor \lozenge q)$ is a theorem of $K$.

9. Show that $p \rightarrow \lozenge p$ is a theorem of $T$.

10. Show that $\lozenge p \rightarrow \lozenge \lozenge \lozenge p$ is a theorem of $S4$.

11. Show that $\lozenge(p \rightarrow q) \rightarrow (\lozenge p \rightarrow \lozenge q)$ is a theorem of $K$.

12. Show that $\lozenge(\lozenge p \rightarrow q) \rightarrow (p \rightarrow \lozenge q)$ and $\lozenge(p \rightarrow \lozenge q) \rightarrow (\lozenge p \rightarrow q)$ are theorems of $S5$.

13. Show that adding the schema $\lozenge A \rightarrow \lozenge \lozenge A$ to $K$ has the same effect as adding the schema $\lozenge \lozenge A \rightarrow \lozenge A$ to $K$ (i.e. derive each schema from the other).

14. Show that adding the schema $\lozenge A \rightarrow \lozenge A$ to $K$ has the same effect as adding the schema $\lozenge(A \lor B) \rightarrow (\lozenge A \lor \lozenge B)$. 