Consider the following modal language L:

$$A ::= p \mid \neg A \mid B \to C \mid B \wedge C \mid B \vee C \mid \Box A \mid \Diamond A$$

1. What does the symbol p represent in the description of L above?
2. A *Kripke frame* (W, R) has two components. What are W and R?
3. A *Kripke model* for L consists of a Kripke frame and a valuation v. What is v?
4. If $\mathcal{F}$ is a Kripke frame and v is a valuation, state the meaning of:
   (a) $\mathcal{F} \vDash_w^v A$
   (b) $\mathcal{F} \vDash^v A$
   (c) $\mathcal{F} \vDash A$
5. (a) Characterise the conditions under which $\mathcal{F} \vDash_w^v p$ holds for any atomic formula p of L.
   (b) Characterise the conditions under which $\mathcal{F} \vDash_w^v A \wedge B$ and $\mathcal{F} \vDash_w^v A \to B$ hold for any formulae A and B
   (c) Characterise the conditions under which $\mathcal{F} \vDash_w^v \Box A$ and $\mathcal{F} \vDash_w^v \Diamond A$ hold for any formula A.
6. (a) Describe a Kripke model in which $\Box p \to \Box\Box p$ is false at some world.
   (b) Describe a Kripke model in which $\Box p \to \Diamond p$ is false at some world.
   (c) Describe a Kripke model in which $\Box(A \vee B) \to (\Box A \vee \Box B)$ is false at some world.
   (d) Argue that $\mathcal{F} \vDash \Diamond(A \vee B) \leftrightarrow (\Diamond A \vee \Diamond B)$ for any Kripke frame $\mathcal{F}$.
   (e) Argue that, if $\mathcal{F}$ is transitive, then $\mathcal{F} \vDash \Diamond\Diamond p \to \Diamond p$.
   (f) Describe a condition on the accessibility relation of a Kripke frame $\mathcal{F}$ under which $\mathcal{F} \vDash \Diamond p \to \Box p$.
7. Describe classes of Kripke frames that can be used to characterise the modal logics T, $S4$ and $S5$
8. Suppose that w is a world in a Kripke frame $\mathcal{F}$ and that, for any formula A, $\mathcal{F} \vDash A \to \Box\Diamond A$. Let v be any valuation such that $v(p) = \{w\}$, argue that if wRw' then $w'Rw$. Now conclude that $\mathcal{F}$ is symmetric.
9. (a) Let $\mathcal{F}$ be a frame such that, for any w_1, w_2, if $w_1 R w_2$ then there is no w such that $w_2 R w$. Argue that $\mathcal{F} \vDash \Box\Box(A \wedge \neg A)$.
   (b) Describe a Kripke model in which $\mathcal{F} \vDash \Box(A \wedge \neg A)$.

The theorems of the model logic K are axiomatised by the schemas:

$$(K) \quad \begin{array}{l} \textit{Any Tautology} \\ \Box(A \to B) \to (\Box A \to \Box B) \\ \Diamond A \leftrightarrow \neg\Box\neg A \end{array} \qquad \frac{A \quad A \to B}{B}\ (MP) \quad \frac{A}{\Box A}\ (N)$$

1. What is an axiom schema?
2. Suppose B is a tautological consequence of $A_1 \ldots A_n$, and suppose each A_i is a theorem of K. Show that B is also a theorem of K.
3. Explain why the rule (N) does *not* imply that $p \to \Box p$ is a theorem of K.
4. Write down additional axioms for the logics $T, S4$ and $S5$.
5. Suppose $A \to B$ is a theorem of K (for some A and some B), show that so is $\Box A \to \Box B$ and $\Diamond A \to \Diamond B$
6. (a) Explain why $p \wedge q \to p$ is a theorem of K.
   (b) Show that $\Box(p \wedge q) \to \Box p$ is a theorem of K.
   (c) Conclude that $\Box(p \wedge q) \to (\Box p \wedge \Box q)$ is a theorem of K.
7. Show that $(\Box p \wedge \Box q) \to \Box(p \wedge q)$ is a theorem of K.
8. Show that $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$ is a theorem of K.
9. Show that $p \to \Diamond p$ is a theorem of T.
10. Show that $\Box p \to \Box\Diamond\Box p$ is a theorem of $S4$.
11. Show that $\Box(p \to q) \to (\Diamond p \to \Diamond q)$ is a theorem of K.
12. Show that $\Box(\Diamond p \to q) \to (p \to \Box q)$ and $\Box(p \to \Box q) \to (\Diamond p \to q)$ are theorems of $S5$.
13. Show that adding the schema $\Diamond A \to \Diamond\Diamond A$ to K has the same effect as adding the schema $\Box\Box A \to \Box A$ (i.e. derive each schema from the other).
14. Show that adding the schema $\Diamond A \to \Box A$ to K has the same effect as adding the schema $\Box(A \vee B) \to (\Box A \vee \Box B)$.