Theories Lecture 1

Axiomatic Theories

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What is this Course?

- This course is designed to be a first taster in the philosophy of mathematics
- It will introduce you asking philosophical questions about modern mathematics
- It is *not* a technical course with problem sheets; it is a normal philosophy course
- But doing philosophy of maths requires some technical background; we assume familiarity with Tim Button's *forallx* and *metathory*
- This week will introduce the key mathematical concepts; next week we will see how they work in application to geometry
- The final two weeks will focus on the controversy with Frege and Hilbert. This is where the real philosophical interest lies

- Plan for lectures:

Lecture 1 Axiomatic Theories

Lecture 2 Theories in Action: Geometry

Lecture 3 Philosophy of Geometry

Lecture 4 The Frege–Hilbert Controversy

Different senses of ‘theory’

- Something too abstract to work in reality

  “Free market capitalism works *in theory*”
- A general body of respectable knowledge

  “The *theory* of natural selection”
- A philosophical account of some phenomenon

  ”Utilitarianism is the best moral theory”
- We’ll be concerned with something more formal

Formal Theories

- We're concerned with *Formal* Theories
- A formal theory has three main ingredients:
  1. A *formal language*, like the language of FOL
  2. A *deductive system*, like natural deduction
  3. A set of sentences, the theory itself
- This is a very specialized sense of the term 'theory', though related to the informal senses above
- But what is the point of studying these things?

Rigour

- In a formalized theory we specify *exactly* what we are assuming
- The logic tells us *exactly* what follows from the assumptions
- The language ensures the whole process is free of ambiguity
- As with all formal matters, we trade nuance for precision and rigour
- These kinds of theory are extremely important in mathematics
- But wholly inappropriate for, e.g. literary theory

Sentences

- In modern logic, we take the elements of a theory to be *sentences*, rather than propositions, statements beliefs etc.
- There are good reasons for doing this - the sentences of a formal language are *very* well behaved
- As usual, we have to gloss over some issues:
  1. Sentences are sensitive to their language, but theories typically aren't. E.g. You can study the theory of evolution in Arabic, it isn't unique to English
  2. Sentences aren't true or false *as such*, they need an interpretation (in the logical sense)
  3. Sentences do not have a unique interpretation, e.g. '$\exists x Fx$'
- We'll set these issues aside for now. They will prove crucial in lectures 3 and 4

The Role of Logic

- A formal theory is a set of sentences in a formal language
- But we also need a *logic* in order to tell us exactly what our axioms entail
- Two relevant senses of ‘entailment’ here:
  1. Deductive entailment, symbolised as $\vdash$
  2. Logical Consequence, symbolised as $\models$
- Our concern is exclusively with the former
- Our definition requires that a theory is *closed* under a deduction relation
- Where $\mathbf{T}$ is a theory, this means that if $\mathbf{T} \vdash \phi$ then $\phi \in \mathbf{T}$
- Intuitively, it means that the theory consists of the sentences we ‘start’ with, plus whatever can be deduced from them by applying the deductive rules over and over again.
- Our concern will usually be with ‘first order’ theories, where the logic is FOL

Example

- Consider the set of sentences $\mathbf{S} = \{$‘It is cold’, ‘It is wet’$\}$
- This is *not* a theory: The sentences are English, and so do not belong to a formal language
- Moreover, the set is *not* closed under the deduction relation that we want. E.g. the sentence ‘It is cold and it is wet’ is not a member of **S**, even though we can deduce it from the members of **S** by conjunction introduction

Example

- We could turn **S** into a theory by formalizing the sentences as P and Q and then closing the set under the deduction relation of TFL, by adding $P \wedge Q$, $P \vee \neg P$ and so on
- Notice that every TFL tautology would need to be added to **S**: a tautology follows from *nothing*, so it certainly follows from **S**
- So where the logic is TFL, FOL, or any other interesting system, the theory will be infinite
- The sentences of a theory that are deduced from the initial sentences are called *theorems*
- There will typically be infinitely many theorems, but the number of axioms can be finite or infinite

Summary

- In English, there are many senses of the word ‘theory’
- We’ve sacrificed nuance for precision, and are exclusively concerned with formal theories
- In our sense, a theory is set of sentences of a formal language closed under some specified deduction relation

Formal Languages

- Let's look in more detail at the specific aspects of our definition of a theory
- A theory must be a set of sentences of a *formal* language
- This means that the symbols and syntax of the language are specified so as to give each sentence a unique reading
- You've already encountered several formal languages; particularly important is the language of FOL

Formal Languages

- A formal language is any language with a fixed (though perhaps infinite) vocabulary, and a syntax that determines exactly what a sentence is, and which symbol is the main connective
- When we study theories, we are generally only concerned with sentences *as such*; we aren't bothered about their meaning
- An exception to this is the logical vocabulary. The meaning of the logical symbols must remain absolutely fixed, no matter how we interpret the names, variables and predicates of the language
- This is the same as it was in *forallx*: No matter how you interpret the rest, the logical vocabulary has a fixed meaning given by a set of truth conditions

The Deductive System

- A theory is a set of sentences in a formal language closed under a deduction relation
- This deduction relation, $\vdash$ must be specified in advance, typically by a set of rules
- The natural deduction system from *forallx* is an example of such a system
- Any set of rules will do. Some are just more interesting!
- These rules aren't concerned with the meanings of the sentences in question. Strictly speaking, they just tell you what you can write down given what you've written down beforehand
- Warning! We're using the term 'theorem' to mean certain members of a theory. Some writers speak of a *logic* as having theorems (namely those formulae derivable from no assumptions)

The Two Turnstiles

- Where ϕ and ψ are sentences and Γ is a set of sentences:
  1. $\Gamma \vdash \phi$ means there is a proof of ϕ from (undischarged) assumptions in Γ
  2. $\phi \vdash \psi$ means that there is a proof of ψ assuming ϕ
  3. $\vdash \phi$ means that there is a proof of ϕ from no assumptions
- Contrast this with the following:
  1. $\Gamma \models \phi$ means that every interpretation which makes every member of Γ true makes ϕ true
  2. $\phi \models \psi$ means that ψ is a logical consequence of ϕ
  3. $\models \phi$ means that ϕ is a logical truth
- The contrast is *critical* to understanding the mathematics and philosophy of theories

Axioms

- Some theories have a privileged subset, the members of which are called *axioms*
- Intuitively, they are the basic assumptions made about the subject matter; they require no proof
- E.g. a formalization of "For every number, there is exactly one that comes immediately after it"
- Perhaps because they strike us as 'obvious', or perhaps we accept them for more pragmatic reasons
- Axioms have an important *epistemological* role in theories
- Interesting theories are infinite: Even with one axiom, P, a theory in TFL will be infinite, since from P we can deduce $P \vee \phi$ for any sentence ϕ
- Axioms are what allow finite creatures like us to deal with these infinitely large sets of sentences

Axiomatic Theories

- We need a formal version of this notion
- We'll only count a subset of a theory as a set of axioms if:
  1. They are *finitely specifiable*
  2. They are adequate to the theory
- Let's look at each condition in more detail

Finitely Specifiable

- If the axioms of a theory are to be any more epistemologically tractable than the theory itself, they must be finitely specifiable
- This means that we can determine, of any sentence in the language, whether or not it is an axiom in a finite amount of time
- This can occur in two ways
- Firstly, the list of axioms might be finite. Then we can simply check, of any sentence, whether it is on the list
- Alternatively, we might have a *rule* that allows us to check whether a sentence is an axiom or not.

Finitely Specifiable

- We need such a rule if a theory has infinitely many axioms; this can happen when a theory includes an axiom *schema*.
- A schema is not a sentence in the language, it is specified using metalinguistic variables, and it tells you that any formula of a particular logical shape is an axiom
- For example, arithmetic includes the axiom schema of *induction*:
- $(\Phi(0) \wedge \forall n(\Phi(n) \rightarrow \Phi(n+1))) \rightarrow \forall n\, \Phi(n))$
- The intuitive principle is that if Φ is true of 0, and where Φ is true of one number, it's true of the next, then Φ is true of all numbers
- But we can't quantify over 'properties' of numbers in FOL. So we have to say in English that we have one induction axiom in the language corresponding to each instance of the intuitive principle

Adequate to the Theory

- An axiom system must be finitely specifiable, or else we can't do anything with it
- But it must also be fit for purpose; that is it must deliver all the sentences of the theory
- In other words, the deductive closure of the set of axioms must be the theory itself
- We'll call a theory $\mathbf{T}$ *axiomatizable* if some finitely specifiable subset Σ is such that $\Sigma \vdash \phi \leftrightarrow \phi \in \mathbf{T}$, for any ϕ
- Some theories have more than one set of axioms, and some have none

Summary

- A formal theory is a set of sentences in a formal language closed under a deduction relation
- A formal theory is axiomatizable if it is identical to the deductive closure of a finitely specifiable subset of itself
- An axiomatic theory is one where the axioms actually *have been* specified.
- Axiomatic theories, particularly of arithmetic and geometry, are of great mathematical and philsophical importance
- We'll finish by discussing some key properties of axiomatic theories

Consistency

- This might be the most important property an axiomatic theory can have
- A theory is consistent iff it doesn't entail a contradiction
- Thanks to EFQ, inconsistent theories entail *every sentence of the language*
- This is not to be confused with *satisfiability* (a.k.a. 'semantic consistency')
- A set of sentences is *satisfiable* iff there is some interpretation under which all of its members are true

Logic vs Theory

- You saw in *Metatheory* that FOL is *complete*, i.e. if $\Gamma \models \phi$ then $\Gamma \vdash \phi$
- This is equivalent to the following condition: if Γ is consistent then Γ is satisfiable
- But do not confuse consistency with satisfiability; they are as different as as $\vdash$ and $\models$
- The completeness theorem is a useful way of proving consistency results. But it is only applicable in the case of first order theories.

Negation Completeness

- People also talk about theories being *complete*. This is not to be confused with being a theory with a complete logic
- What they usually mean is *negation completeness*
- A theory **T** is negation complete iff for any ϕ in the language of the theory:

$$\mathbf{T} \vdash \phi \text{ or } \mathbf{T} \vdash \neg\phi$$

- Thanks to Gödel's theorems, mathematically interesting theories are not usually negation complete

Semantic Completeness

- Other writers talk about theories being 'semantically complete'. Again, this is a different notion
- A theory $\mathbf{T}$ is semantically complete iff there is no ϕ in the language such that both $\mathbf{T} + \phi$ and $\mathbf{T} + \neg\phi$ are satisfiable
- Again, in the first order case only, semantic completeness and negation completeness coincide, thanks to the completeness theorem.
- But again, negation completeness and semantic completeness aren't the same!
- Terminology warning: 'Semantic Completeness' is an unusual term. It's more common to say that that $\mathbf{T}$ has *elementary equivalent models*, where a model of a theory is an interpretation that satisfies it

Conclusion

- In the logician's sense, a theory is a set of formal sentences closed under deduction
- Such theories are often infinite, as a result of the logic behind them
- For an infinite theory to be useful, it needs to be axiomatized, that is it should be the deductive closure of a finitely specifiable set of *axioms*
- Theories are strictly speaking uninterpreted, except for the logical terms
- Consistency is a critical property of theories
- The properties of the logic, like completeness, are separate from the properties of the theory

Next Time on *Theories*

- Next time we'll look at some specific case studies of theories, mainly from geometry
- We'll look at the concept of *independence*, and what sort of questions people ask about theories
- We'll see how to answer questions of consistency and independence
- Then we'll be ready to start asking philosophical questions about theories and axioms: How do we select axioms for the theory of a given subject matter? Are axioms self-evident truths? Must they be true at all?

Reading List

- Wilder, R.L. (1965) *Introduction to the Foundations of Mathematics* 2nd Ed. Chapters 1 & 2
- Barker, S.F. (1964) *Philosophy of Mathematics*. Chapters 2 & 3
- Potter, M.D. (2004) *Set Theory and its Philosophy*. Section 1.1
- Frege, G. (1980) *Philosophical and Mathematical Correspondence*, ed. G. Gabriel. Frege-Hilbert correspondence and Frege-Liebmann correspondence
- Dummett, M. (1996) "Frege on the Consistency of Mathematical Theories" in *Frege and Other Philosophers*
- Blanchette, P. (1996) "Frege and Hilbert on Consistency". *The Journal of Philosophy* 7 317-336