1B Logic: Theories

Introduction to Axiomatic Theories

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What is this Course?

- This course is designed to be a first taster in the philosophy of mathematics
- It will introduce you asking philosophical questions about modern mathematics
- It is *not* a technical course with problem sheets; it is a normal philosophy course
- But doing philosophy of maths requires some technical background; we assume familiarity with Tim Button's *forallx* and *metathory*
- This week will introduce the basic idea of an axiomatic theory, and next week we'll look at some key mathematical properties of such theories.
- The final two weeks will focus on the controversy with Frege and Hilbert. This is where the real philosophical interest lies

- Plan for lectures:

Lecture 1 Introduction to Axiomatic Theories

Lecture 2 Consistency, Completeness, and Independence

Lecture 3 Philosophy of Geometry

Lecture 4 The Frege–Hilbert Controversy

Different senses of ‘theory’

- Something too abstract to work in reality

  “Free market capitalism works *in theory*”
- A general body of respectable knowledge

  “The *theory* of natural selection”
- A philosophical account of some phenomenon

  ”Utilitarianism is the best moral theory”
- We’ll be concerned with *mathematical* theories

Geometry

- Geometry is the mathematical theory of *space*
- It is also one of the oldest academic disciplines, being known in ancient Mesopotamia, Egypt, India, and China. Ahmes, author of the *Rhind Papyrus* (c.1650 BC), is the earliest mathematician known by name
- Geometry was originally developed to assist in the solutions of astronomical, agricultural, and administrative problems
- But in ancient Greece, it began development into the more familiar axiomatic form familiar today

Euclidean Geometry

- The crowning achievement of Greek geometry was Euclid's *Elements* (c.300 BC)
- There Euclid presented an *axiomatic* system for two dimensional geometry with just five basic assumptions, known as axioms (also known as 'postulates')
- The theory known as *Euclidean Geometry* consists of these five basic assumptions, and everything that can be deduced from them
- Though you may not have seen it in axiomatic form before, Euclidean geometry is just the standard 2D geometry you will have seen at school

Euclid's Postulates

1. For any points p and q, exactly one line segment can be drawn connecting them
2. Any line segment can be extended indefinitely
3. For any points p and q exactly one circle can be drawn passing through q with a centre p
4. Any right angles A and B are congruent
5. If a line l passes through distinct lines m and n such that the interior angles between m and l and n and l sum to less than two right angles on one side, then if extended indefinitely m and n will meet on that side

Common Notions

- How do we work out what follows from these axioms?
- Euclid did not have a formal logic like we do, instead he had five ‘common notions’
- These are something like logical principles, stating that identity is transitive, the whole is less than the part, etc.
- In modern geometry, this information is given by a deductive system (like natural deduction from last year)
- Confusingly, some historical works refer to common notions as ‘axioms’ and axioms as ‘postulates’, so be careful!

Definitions

- Euclid's system has one other important component, *definitions*
- Definitions come in two distinct varieties, which it is *absolutely essential* to keep separate in your mind
- Some definitions are of a familiar sort; they are definitions in the language of the theory, which allow you to systematically replace one string of symbols by another
- E.g. '$\exists\alpha\Phi =_{df} \neg\forall\alpha\neg\Phi$'
- A geometric example might be 'A circle is a single-line bounded figure such that for some unique point not on the circle, *c* (for centre) all lines from *c* to any point on the circle are equal'

Primitive Terms

- Not all definitions are like this though; some do not belong to the language of the theory, and do not allow any systematic replacement of expressions
- Rather, they serve to elucidate the meanings of the primitive expressions of the language of the theory
- E.g. ‘A point is that of which there is no part’
- The primitive expressions are those non-logical expressions we use to lay down the axioms of the theory. For instance in geometry, we might choose ‘point’, ’line’, ’between’ ’incident’ and ‘congruent’ (this is Hilbert’s system, he also adds ‘plane’ for the 3D case)
- Choices of primitive terms isn’t unique, but the axioms may need to be altered (e.g. Veblen’s geometry uses only ‘point’, ‘between’ and ‘congruent’)

Interpretation

- The essential point is that, for the purposes of the theory, primitive terms are *undefined* and *unanalysable*. None of Euclid's axioms tell you what a point or a line really is
- All we can do is gesture towards their intended meaning
- Some theories have an *intended interpretation*. This is the interpretation (domain, assignment of names, extensions of predicates etc.) in which axioms are supposed to be true. E.g. arithmetic's intended interpretation has numbers as the domain, '0' is assigned to zero, '+' is assigned the addition function etc.
- In the geometric case, the intended interpretation will be some kind of *space*, perhaps an abstract mathematical one, or perhaps material space itself (more on this later)
- Some theories, notably *algebraic* theories, have no intended interpretation (e.g. Group theory)

Gaps in Euclid

- Although it was held up as *the* gold standard in rigour for about 2000 years, not all of Euclid's deductions are formally valid
- This is because certain proofs (including the first one in the book!) smuggle in facts about the intended interpretation by using diagrams
- Everything Euclid proves is true in the intended interpretation, but not all of it follows from the axioms by logic alone
- That's why when we speak of 'Euclidean geometry' today, we generally mean (something equivalent to) the tidied up version from Hilbert's *Die Grundlagen der Geometrie*, which fill in the gaps

Formal Theories

- To make sure there are no gaps in the proofs, modern mathematics uses *Formal* Theories
- A formal theory has three main ingredients:
  1. A *formal language*, like the language of FOL
  2. A *deductive system*, like natural deduction
  3. A set of sentences, the theory itself
- This is a very specialized sense of the term ‘theory’, though related to the informal senses above
- But what is the point of studying these things?

Rigour

- In a formalized theory we specify *exactly* what we are assuming
- The logic tells us *exactly* what follows from the assumptions
- The language ensures the whole process is free of ambiguity
- As with all formal matters, we trade nuance for precision and rigour
- These kinds of theory are extremely important in mathematics
- But wholly inappropriate for, e.g. literary theory

Sentences

- In modern logic, we take the elements of a theory to be *sentences*, rather than propositions, statements beliefs etc.
- There are good reasons for doing this - the sentences of a formal language are *very* well behaved
- As usual, we have to gloss over some issues:
  1. Sentences are sensitive to their language, but theories typically aren't. E.g. You can study geometry in any natural language, it isn't unique to English
  2. Sentences aren't true or false *as such*, they need an interpretation (in the logical sense)
  3. Sentences do not have a unique interpretation, e.g. '$\exists x F x$'
- We'll set these issues aside for now. They will prove crucial in lectures 3 and 4

The Role of Logic

- A formal theory is a set of sentences in a formal language
- But we also need a *logic* in order to tell us exactly what our axioms entail
- Two relevant senses of 'entailment' here:
  1. Deductive entailment, symbolised as $\vdash$
  2. Logical Consequence, symbolised as $\models$
- Our concern is almost exclusively with the former
- Our definition requires that a theory is *closed* under a deduction relation
- Where **T** is a theory, this means that if $\mathbf{T} \vdash \phi$ then $\phi \in \mathbf{T}$
- I.e. the theory consists of the sentences we start with, (e.g. Euclid's axioms) plus whatever can be deduced from them by applying the deductive rules over and over again.
- Our concern will usually be with 'first order' theories, where the logic is FOL

Example

- Consider the set of sentences $\mathbf{S} = \{$‘It is cold’, ‘It is wet’$\}$
- This is *not* a theory: The sentences are English, and so do not belong to a formal language
- Moreover, the set is *not* closed under the deduction relation that we want. E.g. the sentence ‘It is cold and it is wet’ is not a member of **S**, even though we can deduce it from the members of **S** by conjunction introduction

Example

- We could turn **S** into a theory by formalizing the sentences as P and Q and then closing the set under the deduction relation of TFL, by adding $P \wedge Q$, $P \vee \neg P$ and so on
- Notice that every TFL tautology would need to be added to **S**: a tautology follows from *anything*, so it certainly follows from **S**
- So where the logic is TFL, FOL, or any other interesting system, the theory will be infinite
- The sentences of a theory that are deduced from the initial sentences are called *theorems*
- There will typically be infinitely many theorems, but the number of axioms can be finite or infinite

Formal Languages

- Let's look in more detail at the specific aspects of our definition of a theory
- A theory must be a set of sentences of a *formal* language
- This means that the symbols and syntax of the language are specified so as to give each sentence a unique reading
- You've already encountered several formal languages; particularly important is the language of FOL
- Euclid's axioms are easily formalized in this language

Formal Languages

- A formal language is any language with a fixed (though perhaps infinite) vocabulary, and a syntax that determines exactly what a sentence is, and which symbol is the main connective
- When we study theories, we are generally only concerned with sentences *as such*; we aren't bothered about their meaning
- An exception to this is the logical vocabulary. The meaning of the logical symbols must remain absolutely fixed, no matter how we interpret the names, variables and predicates of the language
- This is the same as it was in *forallx*: No matter how you interpret the rest, the logical vocabulary has a fixed meaning given by a set of truth conditions

The Deductive System

- A theory is a set of sentences in a formal language closed under a deduction relation
- This deduction relation, $\vdash$ must be specified in advance, typically by a set of rules
- The natural deduction system from *forallx* is an example of such a system
- Any set of rules will do. Some are just more interesting!
- These rules aren't concerned with the meanings of the sentences in question. Strictly speaking, they just tell you what you can write down given what you've written down beforehand
- Warning! We're using the term 'theorem' to mean certain members of a theory. Some writers speak of a *logic* as having theorems (namely those formulae derivable from no assumptions)

The Two Turnstiles

- Where ϕ and ψ are sentences and Γ is a set of sentences:
  1. $\Gamma \vdash \phi$ means there is a proof of ϕ where the only (undischarged) assumptions are in Γ
  2. $\phi \vdash \psi$ means that there is a proof of ψ assuming ϕ
  3. $\vdash \phi$ means that there is a proof of ϕ from no assumptions
- Contrast this with the following:
  1. $\Gamma \models \phi$ means that every interpretation which makes every member of Γ true makes ϕ true
  2. $\phi \models \psi$ means that ψ is a logical consequence of ϕ
  3. $\models \phi$ means that ϕ is a logical truth
- The contrast is *critical* to understanding the mathematics and philosophy of theories

Axioms

- Some theories have a privileged subset, the members of which are called *axioms*
- Intuitively, they are the basic assumptions made about the subject matter; they require no proof
- E.g. a formalization of Euclid's axioms
- Perhaps because they strike us as 'obvious', or perhaps we accept them for more pragmatic reasons
- Axioms have an important *epistemological* role in theories
- Interesting theories are infinite: Even with one axiom, P, a theory in TFL will be infinite, since from P we can deduce $P \vee \phi$ for any sentence ϕ
- Axioms are what allow finite creatures like us to deal with these infinitely large sets of sentences

Axiomatic Theories

- We need a formal version of this notion
- We'll only count a subset of a theory as a set of axioms if:
  1. They are *finitely specifiable*
  2. They are adequate to the theory
- Note that we aren't presupposing anything philosophically contentious here
- Let's look at each condition in more detail

Finitely Specifiable

- If the axioms of a theory are to be any more epistemologically tractable than the theory itself, they must be finitely specifiable
- This means that we can determine, of any sentence in the language, whether or not it is an axiom in a finite amount of time
- This can occur in two ways
- Firstly, the list of axioms might be finite. Then we can simply check, of any sentence, whether it is on the list
- Alternatively, we might have a *rule* that allows us to check whether a sentence is an axiom or not.

Finitely Specifiable

- We need such a rule if a theory has infinitely many axioms; this can happen when a theory includes an axiom *schema*.
- A schema is not a sentence in the language, it is specified using metalinguistic variables, and it tells you that any formula of a particular logical shape is an axiom
- For example, arithmetic includes the axiom schema of *induction*:
- $(\Phi(0) \wedge \forall n(\Phi(n) \rightarrow \Phi(n+1))) \rightarrow \forall n\ \Phi(n))$
- The intuitive principle is that if Φ is true of 0, and such that if Φ is true of one number, it's true of the next, then Φ is true of all numbers
- But we can't quantify over 'properties' of numbers in FOL. So we have to say in English that we have one induction axiom in the language corresponding to each instance of the intuitive principle

Adequate to the Theory

- An axiom system must be finitely specifiable, or else we can't do anything with it
- But it must also be fit for purpose; that is it must deliver all the sentences of the theory
- In other words, the deductive closure of the set of axioms must be the theory itself
- We'll call a theory **T** *axiomatizable* if some finitely specifiable subset Σ is such that $\Sigma \vdash \phi$ if and only if $\phi \in \mathbf{T}$, for any ϕ
- Some theories have more than one set of axioms, and some have none

Summary

- A formal theory is a set of sentences in a formal language closed under a deduction relation
- A formal theory is axiomatizable if it is identical to the deductive closure of a finitely specifiable subset of itself
- An axiomatic theory is one where the axioms actually *have been* specified.
- Axiomatic theories, particularly of arithmetic and geometry, are of great mathematical and philsophical importance

Next Time on *Theories*

- Next time we'll look at some key mathematical properties of theories, mainly using examples from geometry
- We'll look at what sort of questions mathematicians ask about theories, and sketch how they are answered
- Then we'll be ready to start asking philosophical questions about theories and axioms: How do we select axioms for the theory of a given subject matter? Are axioms self-evident truths? Must they be true at all?

Reading List

- Wilder, R.L. (1965) *Introduction to the Foundations of Mathematics* 2nd Ed. Chapters 1 & 2
- Barker, S.F. (1964) *Philosophy of Mathematics*. Chapters 2 & 3
- Potter, M.D. (2004) *Set Theory and its Philosophy*. Section 1.1
- Frege, G. (1980) *Philosophical and Mathematical Correspondence*, ed. G. Gabriel. Frege-Hilbert correspondence and Frege-Liebmann correspondence
- Dummett, M. (1996) "Frege on the Consistency of Mathematical Theories" in *Frege and Other Philosophers*
- Blanchette, P. (1996) "Frege and Hilbert on Consistency". *The Journal of Philosophy* 7 317-336