1B Logic: Theories

Consistency, Completeness, and Independence

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Last Time on *Theories*

- Mathematical theories consist of basic assumptions like Euclid's postulates, and whatever we can deduce from them
- We need a formal version of this notion, to make sure there are no gaps in our proofs
- Formal theories are sets of sentences in a formal language closed under a deduction relation
- Interesting theories are infinite, and so need to be presented as the consequences of a finitely specifiable (though not necessarily finite) set of *axioms*

Consistency

- This might be the most important property an axiomatic theory can have
- A theory is consistent iff it doesn't entail a contradiction
- Thanks to EFQ, inconsistent theories entail *every sentence of the language*
- This is not to be confused with *satisfiability* (a.k.a. 'semantic consistency')
- A set of sentences is *satisfiable* iff there is some interpretation under which all of its members are true

Inconsistency Proofs

- Proving that an axiom system is inconsistent is a relatively straightforward mathematical problem
- Simply start with a finite subset of the axioms, and derive a contradiction
- Probably the most famous such result is (a version of) Russell's proof that naive set theory is inconsistent:
- Suppose our theory includes the axiom schema $\exists x \forall y (y \in x \leftrightarrow \Phi)$, where y is the only free variable in Φ
- It follows that $\exists x \forall y (y \in x \leftrightarrow y \notin y)$ is an axiom in our system.
- Suppose r is a set satisfying that axiom. Then $r \in r \leftrightarrow r \notin r$

Consistency Proofs

- Positively showing consistency is not so simple; it would require surveying all proofs with premises from amongst the axioms of **T** and showing that none of them end in $\bot$
- Given the infinity of such proofs, this isn't a strategy we can follow
- Instead, we have to give an *interpretation* of the axioms that makes them all true - a 'model'
- In the case of a theory in FOL, this is exactly the method you used last year to show that sets of sentences were mutually consistent
- In other words, specify a domain, an assignment of names, and extensions of predicates according to which all of the axioms are true
- From the truth of the axioms, their consistency follows: All of the axioms are true, and all the rules of FOL are truth-preserving, so all of the theorems are true. Since no contradiction is true, no contradiction is a theorem of **T**

Relative Consistency

- Unfortunately, it is often not possible to give such consistency proofs
- When it comes to serious mathematical theories, like arithmetic and geometry, the truth of the axioms often entails that the domain is infinite
- So the best we can do is interpret the primitives and domain of one mathematical theory as the intended interpretation of a theory in which we are more confident
- This is fine so long as the consistency of the interpreting theory is taken for granted
- For example, if we can show that the dubious theory has an interpretation where the domain is the numbers, the predicates have numerical extensions, and the constants refer to numbers, we'll know that if arithmetic is consistent, then the doubtful theory is also consistent

Hilbert's Consistency Proof

- Hilbert's *Grundlagen* contains a relative consistency proof of Euclidean geometry
- In particular, Hilbert proves that Euclidean geometry (3D and 2D) is consistent if the theory of the real numbers ('analysis') is consistent
- The real numbers are those that represent quantities on a continuous number line. They are the rational numbers (fractions) plus the transcendental numbers (π, e etc.)
- You're probably familiar from school with the idea of treating points as pairs (in 2D) or triples (in 3D) of real numbers. This is the famous 'Cartesian Coordinate' system, that can serve as a domain for a relative consistency proof
- Then the hard work begins: Find an analytic interpretation of the primitive terms (of which there are six in Hilbert's system), and prove that all of the twenty axioms are true under this interpretation

Completeness: Logic vs Theory

- You saw in *Metatheory* that TFL is *sound* and *complete*, i.e. $\Gamma \models \phi$ iff $\Gamma \vdash \phi$
- FOL is sound and complete too
- This is equivalent to the following condition: Γ is consistent iff Γ is satisfiable
- But do not confuse consistency with satisfiability; they are as different as as $\vdash$ and $\models$

Negation Completeness

- People also talk about theories being *complete*. This is not to be confused with being a theory with a complete logic
- What they usually mean is *negation completeness*
- A theory **T** is negation complete iff for any ϕ in the language of the theory:

$$\mathbf{T} \vdash \phi \text{ or } \mathbf{T} \vdash \neg\phi$$

- Thanks to Gödel's theorems, mathematically interesting theories are not usually negation complete

Semantic Completeness

- Other writers talk about theories being 'semantically complete'. Again, this is a different notion
- A theory **T** is semantically complete iff there is no ϕ in the language such that both $\mathbf{T} + \phi$ and $\mathbf{T} + \neg\phi$ are satisfiable
- For theories in a sound and complete logic, semantic completeness and negation completeness coincide
- But again, negation completeness and semantic completeness aren't the same!
- Terminology warning: 'Semantic Completeness' is an unusual term. It's more common to say that that **T** has *elementary equivalent models*, where a model of a theory is an interpretation that satisfies it

Independence

- The notion of *independence* is of vital importance to the concept of a theory
- We say that a formula ϕ is independent of theory $\mathbf{T}$ iff $\mathbf{T} \nvdash \phi$
- Of particular interest is the case when both ϕ and $\neg\phi$ are independent. In this case we say that $\mathbf{T}$ *doesn't decide* ϕ
- Note that $\mathbf{T}$ is consistent iff $\perp$ is independent of it, and that $\mathbf{T}$ doesn't decide ϕ iff $\perp$ is independent of $\mathbf{T} + \phi$ and $\mathbf{T} + \neg\phi$

Independence

- A particularly important case is when we want to know whether one axiom is independent of the others.
- If an axiom isn't independent of the others, then it is *redundant*; everything that can be proved with it in the theory can be proved without it
- Let $\Sigma - A$ be the deductive closure of the result of deleting A from some particular set of axioms Σ
- In general, we want for each axiom A that $\Sigma - A \nvdash A$
- Or equivalently, we require of a theory $\mathbf{T}$ with axioms Σ that for each axiom A, $\Sigma - A + \neg A$ is consistent

Semantic Independence

- Just as consistency has a semantic counterpart (satisfiability), so does independence
- ϕ is semantically independent of **T** iff $\mathbf{T} \not\models \phi$
- Where the logic is sound and complete, as in FOL, **T** doesn't decide ϕ iff $\mathbf{T} + \phi$ and $\mathbf{T} + \neg\phi$ are both satisfiable
- This is extremely useful, as we'll see in a moment
- But as ever, *the syntactic and semantic notions are distinct*

Non-Independence

- To show that ϕ is non-independent of **T**, just prove ϕ using only assumptions from **T**, as normal
- To prove the non-independence of some axiom A from the other axioms of **T** simply 'delete' A and then show that it follows from the remaining axioms
- Though this may be difficult in practice, it is simple in principle
- An example is that in standard set theory, the axiom of pairs (which states that for any two sets a, b the set $\{a, b\}$ exists) can be straightforwardly proved from axioms asserting that the power set of any set exists and that the empty set exists

The Parallels Postulate

- Recall Euclid's fifth axiom, a.k.a. 'the parallels postulate':

  > If a line l passes through distinct lines m and n such that the interior angles between m and l and n and l sum to less than two right angles on one side, then if extended indefinitely m and n will meet on that side

- Confusingly, this doesn't mention parallel lines at all. It is equivalent to *Playfair's Axiom*:

  > For any line l and any point c not on l, there is exactly one line p passing through c and parallel to l

- Playfair's version is the most common in this day and age

The Parallels Postulate

- Historically, the parallels postulate has been regarded as being of a fundamentally different character to Euclid's other axioms
- Axioms 1–4 were seen as being *self-evidently* true, whereas the parallels postulate wasn't
- A traditional view of axioms is that they must be self-evident truths of the subject matter
- But the parallels postulate was generally believed to be true anyway. So very many mathematicians spent very many years trying to *prove* the parallels postulate from the other axioms
- In other words, they wanted a *non-independence* proof

The Parallels Postulate

- But as it turns out, the parallels postulate *is* independent of the other axioms
- This meant that mathematicians who were trying to prove it from the other axioms got very frustrated:
- 'You must not attempt this approach to parallels. I know this way to the very end. I have traversed this bottomless night, which extinguished all light and joy in my life. I entreat you, leave the science of parallels alone... I turned back unconsoled, pitying myself and all mankind. Learn from my example' (Farkas Bolyai to his son János)
- Since the parallels postulate is independent of the other axioms, there are consistent *non-Euclidean geometries*. This is hugely important

Independence Proofs

- Proving independence, like consistency, can be difficult
- If **T** doesn't decide ϕ, then we would have to survey all possible proofs with premises amongst **T** to conclude this
- Even if $\mathbf{T} \vdash \phi$, proving that $\neg\phi$ is independent can't be done syntactically, since this would amount to a consistency proof for **T**
- So again, we have to start reasoning semantically. Generally to show that $\mathbf{T} \nvdash \phi$ we have to find an interpretation in which all of **T** are true, and so is $\neg\phi$
- As with relative consistency proofs, we can only have as much confidence in such results as we do in the consistency of the theory by means of which we describe the interpretation in question

Hyperbolic Geometry

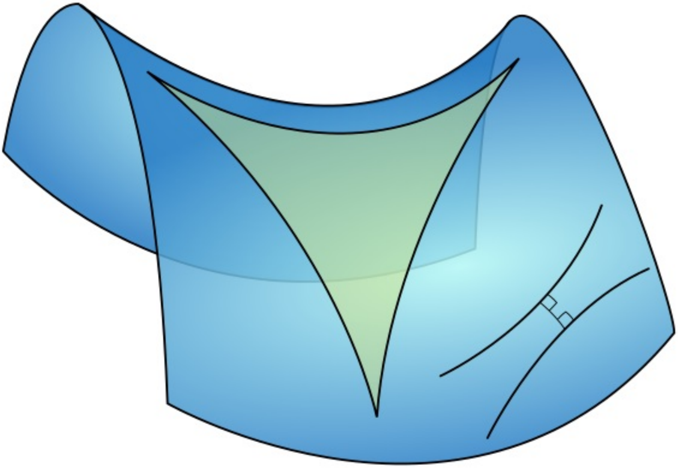
- To show that the parallels postulate is independent of Euclidean geometry, we need an interpretation in which the other axioms are true, and the parallels axiom is false
- To do this, we need to describe another mathematical theory which proves the axioms of Euclidean geometry minus the parallels postulate, and prove that this theory is consistent.
- If the theory doesn't imply the negation of the parallels postulate, there must be a model in which it is false. We'll use *Hyperbolic Geometry*, in which the parallels postulate is replaced by:
- **Hyperbolic Axiom**: For any line l and any point c not on l, there is more than one line p passing through c and parallel to l

Hyperbolic Geometry

- Hyperbolic geometry is highly unintuitive; although it only differs from Euclidean geometry in one axiom, it has many bizarre consequences:
  1. The interior angles of a triangle are always $< 180^o$
  2. All similar triangles are congruent; i.e. there are no triangles of the same shape and different size
  3. The ratio of a circle's circumference to its diameter is $> \pi$
  4. There are no rectangles
- But in light of these consequences, why think that it is consistent?

Hyperbolic Geometry

- Any model of hyperbolic geometry shows that the parallels postulate is independent of Euclidean geometry
- The ingenious argument used by Beltrami is to show that hyperbolic geometry is consistent relative to Euclidean geometry!
- The details of the argument would take us too far afield, but the basic principle is that the surfaces of certain 3D Euclidean constructions are shaped so that they model 2D hyperbolic geometry. Consider the result of inscribing an equilateral triangle on a piece of cloth, then deforming the cloth into a saddle shape:



Summary

- We've seen that most of the important questions about theories revolve around consistency and independence
- Given the infinitary implications of interesting theories, we've seen that the best results of this form that we can get are *relative* to another theory
- Euclidean geometry is consistent relative the theory of the reals
- The parallels postulate is independent of Euclidean geometry if the latter is consistent

Next time on *Theories*

- The technical introduction is over!
- Next time we'll start to focus on the philosophical issues surrounding axiomatic theories
- Some of these are classic philosophical questions: Is there one true geometry? Are the axioms of geometry *a priori*? Are they analytic?
- We'll also start looking at the Frege–Hilbert correspondence, in which these questions are addressed, plus much more besides!