

1B Logic: Theories
Philosophy of Geometry

Wes Wrigley
wdw23@cam.ac.uk

Sidney Sussex College, Cambridge

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Last Time on *Theories*

- ▶ Our technical introduction to theories is now over
- ▶ A theory is a set of sentences closed under a deduction relation
- ▶ The theories we are concerned about are presented as the deductive consequences of axioms
- ▶ Key questions include consistency and independence
- ▶ Euclidean geometry is the axiomatic theory of most historical importance
- ▶ Hilbert proved that it was consistent relative to the theory of reals
- ▶ Beltrami proved that the parallels postulate is independent of the rest of Euclidean geometry
- ▶ This was done by presenting the theory of hyperbolic geometry, which implies all the axioms except the parallels postulate, the negation of which is implied
- ▶ That theory is consistent relative to Euclidean geometry

Today's Lecture

- ▶ Today we'll start looking with a more philosophical eye at axiomatic theories
- ▶ First, there will be an introduction to the pre-19th century orthodoxy on axioms
- ▶ Then I'll introduce the Frege–Hilbert controversy, which is at the center of the modern debate

Axioms: The Big Questions

- ▶ Philosophical questions about axioms have a long history, mostly focusing on the geometrical case
- ▶ Some of the central issues are:
 1. Are axioms true?
 2. If so, what are they true of?
 3. Are they known *a priori* or *a posteriori*?
 4. Are they analytic or synthetic?
 5. Are they necessary or contingent?
- ▶ The answers you might give to these questions are all interrelated; they particularly hinge on the answers to the first two questions

Are Axioms True or False?

- ▶ If you consider axiomatic theories to be the deductive closure of uninterpreted axioms, it seems to follow that they are neither true nor false
- ▶ If I were to write down a bunch of marks on the board which were 'grammatical' according to some rule, but refused to specify an interpretation of them, there would be no more reason to think they were true or false than to think that a teapot or a ceiling is
- ▶ On the other hand, if axioms make assertions about objects, it seems that they must be true or false, just like any other assertion

The Subject-Matter of Geometry

- ▶ The issue is a little clearer if we think about the *subject-matter* of a theory, like geometry
- ▶ If Euclidean geometry is supposed to be about the physical space in which we live, does that help clear up the issue?
- ▶ Not really! Thanks to experiments confirming the physics of general relativity, it is quite common to hear people say things like 'space is non-Euclidean'
- ▶ But interpreting Euclidean geometry as about physical space requires theoretical decisions; for example interpreting the term 'straight line' as meaning the path along which a ray of light travels

The Subject-Matter of Geometry

- ▶ Surprisingly perhaps, the path of light rays is distorted by the presence of massive bodies like stars. So on that interpretation, space really is non-Euclidean
- ▶ BUT it is open to the defender of Euclidean geometry to insist that 'line' is to be interpreted as referring to *lines* not to light rays. According to this hypothetical person, the experiments would simply show that light doesn't travel in straight lines
- ▶ That might make your theoretical physics look untidy, but it isn't incoherent. To establish that, we'd need to show that it was *analytic* or something similar, that light travels in straight lines
- ▶ So the situation isn't as simple as it may at first appear

The Orthodox View

- ▶ Things will be easier if we consider a concrete example, which I'll call the *orthodox view*
- ▶ The view is Kant's, though (some version of) it seems to have been the standard view of mathematicians and philosophers from Plato until the 19th Century
- ▶ According to this view, geometrical axioms are about *space*; 'lines' are to be interpreted as straight lines, 'point' as referring to points in space, etc.
- ▶ The meaning of the primitive terms can be elucidated by Euclid-style "definitions", but our grasp of these terms derives primarily from *intuition*, the faculty of immediate singular representation in space and time of an object to a thinking subject.

The Orthodox View

- ▶ The axioms of Euclidean geometry are *true*, that is they tell us true things about points, lines, shapes etc.
- ▶ The axioms are self-evident and can be known *a priori*. By means of deductive mathematical arguments we can know geometrical theorems *a priori*
- ▶ Being true *a priori* geometrical axioms are *necessarily* true
- ▶ However, because they concern the form to which intuition of objects must conform, they are *synthetic*
- ▶ Some geometrical facts, e.g. 'All triangles are three-sided' are analytic, because the predicate-concept (three-sidedness) is contained in the subject-concept (triangularity). But these are special cases
- ▶ In general, geometric truths are not merely analytic or conceptual, they form part of the structure to which any possible experience must conform

The Orthodox View

- ▶ Of course, philosophers weren't talking about analytic truths before Kant, and mathematicians often don't formulate philosophical thoughts explicitly or precisely
- ▶ Nonetheless, the view is fairly representative of the orthodoxy before the rise of non-Euclidean geometry: Geometry is not a symbol-game, it is the science of space, and Euclid's axioms are the correct account of how it behaves. We can derive further truths *a priori* by deduction, and in virtue of their *a priority*, such truths are necessary
- ▶ Other views that roughly conform to this include Plato, who thought that geometry represented knowledge of certain forms that were eternal, immutable, and known by recollection. But given that Plato predates the axiomatization of geometry by Euclid, we should be careful not to be anachronistic in ascribing views about *axioms* of geometry

Heterodox Views

- ▶ This view was greatly put into question by the rise of non-Euclidean geometries
- ▶ Given the consistency of non-Euclidean geometries like hyperbolic and elliptic geometries, all of the traditional thinking about axioms came into question
- ▶ Firstly, if there are consistent non-Euclidean accounts of space, it seems somewhat dubious to claim *necessity* for the axioms of Euclidean geometry
- ▶ Secondly, it seems like the axioms of geometry can't give us *a priori* knowledge about space; or if it can such knowledge could only be hypothetical. For example, a proof of Pythagoras' theorem only gets us as far as knowing that *if* space is Euclidean *then* the theorem holds

The Pure/Applied View

- ▶ These kinds of worry lead to a view espoused (at least for a time), by Einstein:
- ▶ 'as far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality' (1921, *Geometry & Experience*)
- ▶ On this view, we distinguish between *pure* geometry and *applied* geometry
- ▶ Pure geometry, roughly speaking studies abstract mathematical objects called 'spaces'. Applied geometry studies physical space.
- ▶ The latter study is on a par with any other area of theoretical physics, and as it turns out space isn't Euclidean after all
- ▶ In the sense of a pure geometry, however, Euclidean geometry is perfectly in order

The Pure/Applied View

- ▶ This view allows us to save a lot of traditional thinking about geometry: The axioms of Euclidean geometry can be known *a priori* and express necessary truths, but only about a special kind of object, *Euclidean Spaces*
- ▶ So the element of the traditional view that can't be saved is perhaps *syntheticity*. It is simply part of the concept of a Euclidean space that the axioms of Euclidian geometry hold; that's as trivial as 'all triangles are three-sided'
- ▶ According to this view too, the primitive terms of the theory are not meaningless, though unlike in the orthodox view they are importantly reinterpretable, namely when we want to consider the adequacy of a geometry as applied

Summary

- ▶ We've seen that the main elements of philosophical thinking about geometry concern a number of quite traditional philosophical issues:
 1. Reference/Meaning
 2. Truth
 3. Necessity, *a priori*, analyticity
- ▶ We canvassed the traditional Kantian view, which I called *orthodox*, and a more modern view, the *pure/applied* view
- ▶ For the rest of the course, we'll be concerned with a related, but different debate, namely the *Frege–Hilbert Controversy*
- ▶ Hopefully, a lot of what they discuss will now be familiar. But it's important not to 'project' expectations on to the debate
- ▶ Frege's view is clearly more aligned with the orthodoxy. Hilbert's view is more in line with the pure/applied view. But differences remain!

Background

- ▶ Frege and Hilbert met in 1895, and following this exchanged a series of nine letters discussing philosophical issues surrounding the foundations of geometry
- ▶ What's of particular interest is that Hilbert's *Grundlagen*, which is widely regarded as one of the most significant contributions to geometry ever, is seen by Frege as being both philosophically suspicious *and* mathematically useless
- ▶ One common view is that Frege was wrong and Hilbert was right. Less standard is the view that they were both correct 'in their own terms' and were talking past one another
- ▶ If you only read one thing for this course, it should be the letters between Frege and Hilbert. If you only read two, the other should be Patricia Blanchette's paper 'Frege and Hilbert on Consistency'

Preamble

- ▶ The disagreements between Frege and Hilbert centre around two issues. The division between the two is somewhat artificial, but I think helpful
- ▶ Firstly, the two disagree about the relation between axioms and definitions.
- ▶ Secondly, they disagree about the truth of mathematical statements and the existence of mathematical objects. We'll look at these two next week.
- ▶ But today, we'll take a quick survey of the views of Frege and Hilbert in order to set the disagreement up properly

Hilbert's Basic Conception

- ▶ We'll start with Hilbert's approach to axiomatic theories. It is remarkably close to the modern approach we looked at, because Hilbert was a major player in its development
- ▶ What's largely different is that Hilbert's work pre-dates the rise of model theory, and with it a sharp understanding of semantic notions like completeness, logical consequence and set-models
- ▶ Nonetheless, the syntactic material should be familiar

Hilbert's Theories

- ▶ For Hilbert, a theory is a set of *partially interpreted sentences* closed under a syntactic deduction relation
- ▶ The deduction relation, \vdash , is governed by finitary transformation rules which are purely syntactic. So $\phi \vdash \psi$ *just* means that if you have written down ϕ , then by applying the rules finitely many times, you can write down ψ
- ▶ So axioms in a Hilbertian theory are *meaningless*. They contain uninterpreted parts, namely the primitive terms, and are not 'truth-apt'

Syntactic Consistency

- ▶ A consequence of this view is that Hilbert's notion of a consistent theory is the standard syntactic one; a set of partially interpreted sentences Γ is consistent iff $\Gamma \not\vdash \perp$
- ▶ So a consistency proof for Hilbert is a relative consistency proof
- ▶ A possible exception to all this is some *very* minimal fragment of elementary mathematics. Roughly speaking, Hilbert came to think that the mathematics of finitary syntax was contentual and should be given an absolute consistency proof
- ▶ But we'll set that aside for now, and focus on arithmetic and geometry

Property Consistency

- ▶ Another important concept for Hilbert is *property consistency*
- ▶ Consider the following sets of partially interpreted axioms:
 1. $\{\forall x\forall y(Rxy \rightarrow Ryx); \forall xRxx\}$
 2. $\{\forall x\forall y(Rxy \rightarrow \neg Ryx); \forall xRxx\}$
- ▶ Hilbert thinks of these sets as defining syntactically complex properties
- ▶ Such a property is said to be *consistent* if some concept(s) or sets could have the property so defined
- ▶ In this case, 1. is property consistent. Indeed it is satisfied by any relation that is reflexive and symmetric
- ▶ But 2 is property inconsistent. To satisfy it we'd need a non-empty domain and relation over it which was reflexive *and* asymmetric

Consistency and Independence Proofs

- ▶ Hilbert thus describes axioms as defining a ‘scaffolding of concepts’
- ▶ Once we have syntactic and property consistency, we can likewise define independence; the property defined by ϕ is independent of the property defined by Γ just in case the property defined by $\Gamma + \neg\phi$ is property-consistent
- ▶ A proof that Γ is syntactically consistent is also a proof that the property defined by Γ is property consistent
- ▶ A proof that the property defined by Γ is property consistent is also a proof that Γ is syntactically consistent
- ▶ All such proofs are relative to the consistency of a background theory, but it means that Hilbert has essentially all the modern tools available to prove these results

Summary

- ▶ Hilbert's basic view of axioms is a very modern one
- ▶ Axioms are partially interpreted sentences
- ▶ Their consistency is a syntactic notion
- ▶ They define complex properties with a related consistency notion, which is roughly 'satisfiability relative to a background theory'
- ▶ We'll move on to Frege, starting with a reminder of the Fregean picture of language

Frege: Philosophy of Language

- ▶ Frege's view of language is three-levelled: There is syntax, sense, and reference
- ▶ According to Frege, grammatical strings in a language can be carved up into 'complete' and 'incomplete' parts. Complete expressions are those without 'blank spaces' or variable places. These Frege calls 'names'. Names include traditional singular terms like 'Frege' and '2+2', but also sentences (which Frege takes to be the names of truth values)
- ▶ There are also incomplete expressions, or 'functors', which have one or more blank argument places. These are divided into a hierarchy distinguished by the 'type' of variable place in the expression

Frege: Philosophy of Language

- ▶ A 'first level' functor has a variable place for names, e.g. 'x is wise'
- ▶ A second level functor has a variable place for first level functors, e.g. 'Socrates is F '
- ▶ And so on. In general an $n + 1$ level functor has a variable place for an expression of level n . Names can be considered as expressions of level 0
- ▶ The type of expression constrains the sense of an expression, which in turn determines the reference.

Frege: Philosophy of Language

- ▶ Recall that the sense of an expression is the mode of presentation of its referent. That explains the cognitive difference between 'Hesperus' and 'Phosphorus', even though they have the same referent.
- ▶ The sense of a sentence is called a *thought* and is the mode of presentation of its truth value. Hence why equivalent sentences can have different cognitive value.
- ▶ At the level of reference, expressions of different types always have a different kind of reference, given that the sense of such expressions is of a fundamentally different kind.
- ▶ Names refer to objects, and n -level functors refer to n -level functions
- ▶ A function the value of which is always a truth-value is called a *concept*

Frege on Theories

- ▶ Frege takes the more traditional view that axiomatic geometry has a subject matter, about which the axioms are self-evident truths
- ▶ Hence for Frege, the interest in theories is not limited to syntactic considerations; we can think about what thoughts/propositions follow from what assumptions
- ▶ Derivability is at the level of syntax, and satisfiability at the level of reference, so we need a new notion to operate at the level of sense; lets call this *Frege-Consequence* \models_F
- ▶ This is an entailment relation between interpreted sentences. Similarly, a set of sentences Γ is *Frege-Consistent* iff $\Gamma \not\models_F \perp$, and ϕ is Frege-Independent of Γ just in case $\Gamma \not\models_F \phi$

Conclusion

- ▶ Today we've started to explore central philosophical concepts about axioms
- ▶ The traditional view of arithmetic is Kant's: the axioms are about space, they are necessarily true, and are synthetic *a priori*
- ▶ In the wake of non-Euclidean geometries, a new view developed, which sharply distinguishes a pure geometry from its physical interpretation
- ▶ We've also started looking at the views of Hilbert and Frege
- ▶ Hilbert's view is very closely aligned to the syntactic method of studying theories
- ▶ Frege, on the other hand takes logical notions like consequence, independence, and consistency to hold between *propositions*, which he calls 'thoughts'

Next Time on *Theories*

- ▶ We'll finish off the course with a detailed look at the clash between Frege and Hilbert
- ▶ They chiefly disagree about the relationships between axioms and definitions, and between consistency, truth, and existence
- ▶ I'll argue that, as usual, both were correct in some respects and mistaken in others