1B Logic: Theories

The Frege–Hilbert Controversy

Wes Wrigley
wdw23@cam.ac.uk

Sidney Sussex College, Cambridge

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Last Time on *Theories*

- Last time, we started looking at traditional philosophical thinking about axioms
- We also saw that such thinking was upset by the discovery of non-Euclidean geometries
- A related debate is the Frege–Hilbert controversy
- Last time we saw that Frege has a more traditional view of axioms, and crucially thinks that logical relations hold between *propositions*, which he calls ‘thoughts’
- On the other hand, Hilbert’s concerns were almost purely syntactic, and strongly resemble modern methods of studying theories

Today's Lecture

- First, we'll try to understand why Frege thought Hilbert's methods were useless
- Then we'll get clear on what the philosophical disagreements are between Frege and Hilbert:
  1. Implicit and Explicit Definitions
  2. Consistency and Truth
  3. Consistency and Existence
- We'll also look at whether there is anything of value in Frege's approach

Frege-Consistency and Syntactic Consistency

- Recall that Frege takes consistency to be a property of propositions; they are Frege-consistent if they could all be true together
- Syntactic consistency *does not* show Frege-consistency
- For example, ‘I am an only child’ and ‘I have a sister’ are syntactically consistent, if formalised as sentence letters, but they are Frege-inconsistent
- So although syntactic inconsistency entails Frege-inconsistency, the same is not true for positive consistency claims

Frege-Consequence and Independence

- Similarly for Frege, a thought ϕ is independent of a set of thoughts Γ iff $\Gamma \not\models_F \phi$
- In other words, ϕ is independent if it might be false where all the members of Γ are true
- Frege takes deduction to be a reliable logical notion, i.e. if $\Gamma \vdash \phi$ then $\Gamma \models_F \phi$
- But from $\Gamma \nvdash \phi$ we can't conclude $\Gamma \not\models_F \phi$
- This is because a better formalisation might make a proof possible
- E.g. From 'Cecil is a lion' there is no proof in propositional logic of 'something is a lion'
- But in FOL the proof would be almost trivial!

Consistency and Independence Proofs

- So that's why Frege takes Hilbert's methods to be useless - they don't show Frege-consistency or Frege-independence
- Indeed, for Frege an axiom is a self-evident truth, so there's no need to prove that axioms are consistent with one another
- At this point, we might wonder whether both writers are simply 'correct in their own term'
- Frege and Hilbert are free to use the words 'consistency', 'axiom', and 'definition' in their own ways, and each is right in their own terms
- I don't think this is the whole story, because Frege and Hilbert are operating with wildly different conceptions of what logic is and why it's important. We'll come back to this at the end of the lecture

Hilbert's Thesis

- Hilbert's most contentious views are summed up in the following quotation:

  > *[I]f the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the the things defined by the axioms exist'* – Letter to Frege 29.12.1899

- Frege disagrees with *all* of this. Firstly, that the consistency of the axioms implies their truth. Secondly, that the axioms define anything. Thirdly, that the consistency of the axioms establishes the existence of anything

Implicit Definition

- Hilbert's axioms, recall, are *partially interpreted* sentences. The logical vocabulary has a fixed interpretation, but the primitive terms of the theory are undefined and uninterpreted
- Hilbert thinks, however, that a consistent system of axioms *collectively implicitly defines* the primitive terms that appear in the theory.
- In the geometrical case, this will be *point*, *line* etc.
- When we start looking at the theory, it is not assumed that we know what the primitives mean in advance, and they have no explicit definition in the language of the theory
- Hilbert's idea is that in coming to see that the theory is consistent, we grasp the meaning of the primitive terms

Implicit Definition

- In other words, there is nothing more to *being a point* or *being a line* than is specified in the axioms
- The relations between primitive terms specified by the axioms exhaust their meaning
- So the primitive terms just mean whatever they have to in order to satisfy the axioms
- Inconsistent axiom systems are unsatisfiable, and define nothing
- Consistent systems are satisfiable, and define the ‘scaffolding of concepts’ described by the system (a complex $n-$place property, for a system with n primitive terms)

The Watch Argument

- Frege, however, complains that ‘the axioms are made to carry a burden that belongs to definitions’
- In particular, Frege thinks that *definitions* don’t *assert* anything (as an axiom might), but only serve to *lay down* a policy concerning the use of a symbol
- A consequence of this view is that definitions should be *eliminable*. E.g. suppose I define the existential quantifier as ‘$\neg\forall\alpha\neg\phi$’. If I ever see ‘$\exists\alpha\phi$’ I know how to get rid of the defined symbol
- Consequently, I know I can introduce the defined symbol whenever I see $\neg\forall\alpha\neg\phi$

The Watch Argument

- Frege's complaint is that Hilbert's axioms can't possibly *define* terms like 'point' and 'line' because they do not determine whether, for instance, his pocketwatch is a point
- Frege thinks that axioms must be *true* and hence must be meaningful in advance. Therefore they don't *define* anything. The meaning of the primitive terms can be gestured to in advance, by Euclid-style definitions ("elucidations"), but these are inessential
- That complaint is tied to a specific concept of an axiom that Hilbert doesn't accept
- You might think that the watch argument is similarly tied to a concept of definition that Hilbert just isn't on board with
- That might be too quick though. Frege is correct that Hilbertian definitions don't allow us to sort geometrical objects from non-geometrical objects, which will cause some problems later

The Equations Argument

- But things get worse for Hilbert. Frege offers the following argument:

  'Your system of definitions is like a system of equations with several unknowns, where there remains a doubt... whether the unknown quantities are uniquely determined' – Letter to Hilbert 6.1.1900

- In other words, Frege thinks that a definition must *uniquely* determine what it defines. This can be a plurality though; e.g. 'x *is even* $=_{df} \exists y(x/2 = y)$' is perfectly acceptable
- Hilbert's implicit definitions don't do this, however

The Equations Argument

- Consider the following pair of equations:
  1. $x \times y = 20$
  2. $x + y = 9$
- It could be that $x = 4$ and $y = 5$, or it could be that $x = 5$ and $y = 4$
- For this reason, no one would consider the unknowns here to be *defined* by these equations
- Similarly, thinks Frege, we can't consider primitives to be defined by axioms, since the axioms can be interpreted in several different ways that satisfy them

The Levels Problem

- Both of Frege's previous arguments are intimately tied up with his picture of predicates and concepts as being typed to a *level*
- Hilbert thinks of primitives as functioning like variables, and axioms as defining complex properties where they are consistent
- Crucially, the properties defined by axiom systems are *second level*, they are properties *of properties*
- E.g. the Euclidean axioms can be satisfied in the reals. Here the "variable" predicate 'x is a point' is interpreted as meaning x *is a pair of reals*. And x *is a pair of reals* is a first level concept satisfied by all and only pairs of reals

The Levels Problem

- In other words, consistent sets of axioms allow you to determine when a *concept* is functioning as point-concept or a line-concept
- But a definition of ‘point’ should allow you to determine which objects are points
- Hence why we can’t tell, using Hilbert’s axioms, whether a watch is a point; it is because the axioms don’t uniquely determine a first-level concept
- So Frege’s complaint is ultimately: If axioms define *anything* they define something of the wrong level
- And Frege is also sceptical of whether they can even do that, since they contain the very terms allegedly being defined

Consistency and Truth

- One of Hilbert's other main claims is that *if* an axiomatic theory is consistent *then* the axioms are true
- This is a substantial philosophical thesis. The converse is accepted by Frege, given that it's trivially true
- One thing that's puzzling initially is that Hilbert seems to think that axioms can be both meaningless and true, which is plainly incoherent
- However, we've already seen that for Hilbert, individually meaningless axioms collectively implicitly define the primitives, so there is a sense in which they are meaningful and hence truth-apt
- Perhaps Hilbert is wrong. But Frege thinks that *even if* you set aside the worries about implicit definition, consistency still doesn't entail truth

Consistency and Truth

- Recall that Hilbert's relative consistency proofs follow roughly the pattern of interpreting the primitives one theory in terms of another theory, and showing that all of the axioms are true in the new interpretation. It follows that the axioms contain no contradiction, *if* the interpreting theory is consistent
- Frege concedes this much; he says that 'if the latter is free from contradiction [i.e. the interpreting theory], we can infer the general proposition [schematic axiom being interpreted] is free from contradiction'
- BUT Frege thinks the converse isn't true

The Pythagorean Argument

- Frege offers the ‘Pythagorean’ argument:

  > *Suppose we have proved that in an equilateral rectangular triangle the square on the hypotenuse is twice as large as the square on one of the sides... [w]e can now conclude that the proposition [Pythagoras’ theorem] does not contain a contradiction either within itself or in relation to the geometrical axioms. But can we now conclude further: therefore Pythagoras’ theorem is true?* – Letter to Hilbert 6.1.1900

- The problem that Frege is gesturing at is that, for all we know on the basis of our assumptions, there might be a counterexample to Pythagoras’ theorem in a non-equilateral case

The Pythagorean Argument

- Suppose we grant to Hilbert that the consistency of Euclid's axioms means that they define 'point', 'line' etc.
- From our proof, we know that the axioms are consistent because of their interpretation in the reals
- But might they not still be *false* of points and lines, because of the special properties of the geometric interpretation?
- E.g. P and Q are clearly consistent. Just interpret P as 'It is Monday' and Q as '2+2=4'
- But if we interpret P as 'I have a sister' and Q as 'I am an only child', they are no longer both true, and they cannot be, given this particular interpretation

Consistency and Truth

- How can Hilbert respond to this argument?
- He might say that there's no more to being a point or line that given in the axioms, so by saying that they were true, he meant no more that 'true in the interpretation given'
- But that's trivial! Since the real number interpretation was used to establish the consistency of the axioms, of course the axioms are all true *in that interpretation*
- Frege takes Hilbert to be making the stronger claim that a consistent set of axioms defines the primitives, and simply in virtue of that consistency it is true to say that geometric objects, such as points, lines etc. behave in such-and-such a way
- In other words, the consistency of the theory is sufficient to establish the truth of the axioms in (what Frege might call) the intended interpretation
- But it is hard to see how the behaviour of the reals could tell us this

Consistency and Existence

- The last major claim made in Hilbert's thesis is that the consistency of the axioms suffices for the existence of the objects so defined
- Unsurprisingly, this is a claim Frege takes issue with!
- It is related to the previous argument: If the consistency of the axioms doesn't establish the truth of the geometric axioms, it is hard to see why consistency would entail the existence of geometric objects
- But Frege still has something to say in particular on the matter of existence

The Ontological Argument Argument

- Frege argues that Hilbert's view roughly entails that the ontological argument for the existence of God should work. Frege takes this to be a disastrous consequence
- Take the following sentences:
  1. *A* is an intelligent being
  2. *A* is omnipresent
  3. *A* is omnipotent
- According to Frege, the only way of showing that these properties are mutually consistent is 'pointing out an object that has such properties'
- Hilbert's view, however seems to entail that God exists because Nicki Minaj exists. Just let *A* refer to Nicki, and interpret 'intelligent being' as true of intelligent beings, 'omnipresent' as true of rappers and 'omnipotent' as true of anyone who wrote *Pink Friday*

The Ontological Argument Argument

- Again, there are two ways to interpret Hilbert. We might take him to think that being intelligent, omnipotent and omniscient is simply satisfying the axioms in some interpretation
- In that case, all the example proves is that God exists where 'God' is interpreted as Nicki. But the existence of Nicki *trivially* proves the existence of Nicki
- If we take the stronger sense, that 'God' is implicitly defined by the axioms above, then it looks like Nicki's existence really does entail that God exists, an unwelcome philosophical consequence
- Hilbert stopped replying to Frege's letters after this. But there might be something to be said on his behalf

Hilbert's Reply?

- A common response on Hilbert's behalf is that consistency as a criteria of truth and existence is only supposed to apply to *mathematical objects*, and hence doesn't apply to things like God
- But this won't work as it stands - we can simply add to the God-axioms 'A is a mathematical object', and interpret the new predicate as meaning that x wrote *Roman Reloaded*
- Now, it looks like Nicki proves that God exists and is a mathematical object
- Normally this trick wouldn't work. The constraint on Hilbert's thesis was stated in the *metalanguage*. But 'is a mathematical object' above is in the language of the theory, and thus subject to reinterpretation. So there's no guarantee that it means what we would when we say 'is a mathematical object' in English
- But by Hilbert's own lights, the statement is *true* in the intended interpretation if it's part of a consistent theory, so this standard reply might not be available

Taking Stock

- To my mind, the only fix available to Hilbert would be some division of *language* (rather than objects) into mathematical and non-mathematical. Whether that can be done, I don't know
- It also wouldn't answer Frege's other objections
- We've seen that Hilbert's more extravagant philosophical claims about definition, truth and existence have some serious problems
- But Frege's view certainly has issues too!
- We'll end the series by taking a broader look at the successes and limitations of Hilbert and Frege's positions

Assessing Hilbert

- Hilbert's project of axiomatizing mathematics was a resounding success
- It is more-or-less standard mathematical practice to take theories in a purely syntactic way
- The list of fruitful results is enormous, thought even just the proof of the consistency of Euclidean geometry would be enough to show that Frege's assessment of Hilbert's method as 'useless' is seriously mistaken
- But you might think that Hilbert's more philosophical claims are dubious - it seems highly unlikely that syntactic considerations alone can settle all the issues of definition, truth, and existence in mathematics

Assessing Frege

- By contrast, Frege's naysaying of consistency and independence results would have stunted the development of axiomatic mathematics if it had caught on
- And insisting that axioms must be self-evident truths strikes many readers today as terminological nitpicking
- That said, Frege's conception of logic is *philosophically* important it tells us about:
    1. What we can *know* on the basis of what we've assumed
    2. What existential statements are logical consequences of the theories we accept
    3. The analytic-synthetic distinction
- These are all things philosophers care about, even if they aren't important for the study of axioms in the way Frege thought

Conclusion

- During this course, we've seen how the mathematics of axiomatic theories work today
- We've seen how this can be applied to the case of geometry
- We've explored philosophical thinking about geometry
- We've studied the controversy between Hilbert and Frege on the nature of axioms
- Mostly it's been a first introduction to the philosophy of mathematics, which you'll have the opportunity to explore in greater detail next year