Goedel's Theorem 10

Lecture Contents

- $\mathbf{Con}_T$, a canonical consistency sentence for T
- The Formalised First Theorem
- The Second Theorem
- Why the Second Theorem matters
- What it takes to prove it

Definitions

- $\bot$ is T's absurdity constant if it has one, or else it is an abbreviation for $\overline{0} = s(\overline{0})$.
- $\mathbf{Con}_T$ abbreviates $\neg\mathbf{Prov}_T[\overline{\ulcorner\bot\urcorner}]$
- Note that $\mathbf{Con}_T$ is Π_1.
- There are other ways of formalising consistency statements, but this is the most straightforward.

Formalising the first theorem

- First theorem: If T (axiomatised, containing $\mathbf{Q}$) is consistent then G_T is not provable in T.
- $\mathbf{Con}_T \to \neg\mathbf{Prov}_T[\overline{\ulcorner G_T \urcorner}]$
- But if T is strong enough, we can formalise the reasoning of the first theorem in T (e.g. formalising the informal 'if. . . then. . . ' as $\to$).

Formalising the first theorem

- For strong enough T,

$$T \vdash \mathbf{Con}_T \rightarrow \neg \mathbf{Prov}[\overline{\ulcorner G_T \urcorner}]$$

Call this the *Formalised First Theorem*.

Unprovability of consistency

- But $T \vdash G_T \leftrightarrow \neg\mathbf{Prov}[\overline{\ulcorner G_T \urcorner}]$. So if $T \vdash \mathbf{Con}_T$ then $T \vdash G_T$.
- But if T is consistent then $T \nvdash G_T$.
- So if T is consistent then $T \nvdash \mathbf{Con}_T$.
- The second theorem destroys Hilbert's programme, if his finitary methods are recursively axiomatisable.

Proving the second theorem

- The provability predicate $\mathbf{Prov}_T$ abbreviated $\exists z \mathbf{Prf}_T[z, x_1]$, and is Σ_1.
- So it is reasonable to suppose we need at least induction for Σ_1 wff.
- A theory is Σ-normal [Peter Smith Shorthand], if it is axiomatised, contains $\mathbf{Q}$, and also also includes induction at least for Σ_1 wffs.
- So we might conjecture that if T is Σ-normal, then T proves the formalised First Theorem, and so if T is consistent, $T \nvdash \mathbf{Con}_T$

Box Notation

- To improve readability, and make a connection with modal logic, let us write $\Box_T \phi$ for $\mathbf{Prov}_T[\overline{\ulcorner \phi \urcorner}]$.
- So we can abbreviate the formalised first theorem as:

$$\neg \Box_T \bot \to \neg \Box_T G_T$$

- Drop the subscript for simplicity, so $\Box$ is short for $\Box_T$.

Derivability conditions

- The *derivability conditions* hold in T when, for any ϕ
  1. If $T \vdash \phi$ then $T \vdash \Box\phi$
  2. $T \vdash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
  3. $T \vdash \Box\phi \to \Box\Box\phi$
- If T is Σ-normal, then the derivability conditions hold for T. [tedious proof]
- If T is Σ-normal and the derivability conditions hold for T, then T proves the formalised First Theorem.

Deriving the formalised first theorem

- $T \vdash G_T \to \neg\Box G_T$
- So $T \vdash \Box G_T \to \Box(\Box G_T \to \bot)$
- So $T \vdash \Box G_T \to \Box\bot$
- And the formalised first theorem follows.