Goedel’s Theorem 8
Lecture Contents

- The idea of diagonalisation
- How to construct a ‘canonical’ Goedel sentence
- If an axiomatised theory is sound, it is negation incomplete
- Applying that to \( PA \)
- \( \omega \)-incompleteness, \( \omega \)-inconsistency
- If \( PA \) is \( \omega \)-consistent, it is negation incomplete
- Generalising that result to \( \omega \)-consistent an axiomatised theories which extend \( Q \)
Review

- Fix a suitable Gödel coding of wffs and strings of wffs of $L_A$.
- $Prf_Z(m, n)$ is the relation which holds just if $m$ is the g.n. of a sequence of wffs that is a $Z$ derivation of a sentence with g.n. $n$. This relation is decidable if $Z$ is (recursively) axiomatisable.
- Any recursive function or relation can be expressed by a wff of $L_A$.
- The axiom of induction which strengthens $Q$ to $PA$.
- Any recursive function or relation can be captured in $Q$ and hence in $PA$. 
We will form an $L_{A_1}$ wff $U$ and then substitute its own g.n. $\overline{U}$ into it: $U[\overline{U}]$.

The action of substituting-its-own-g.n. is called *diagonalisation*.

The diagonalisation of $\phi$ is $\phi[\overline{\phi}]$, i.e. $\phi[x_1/\overline{\phi}]$. 
Diagonalisation

- We could also define it as $\exists x_1 (x_1 = \overline{\phi n} \land \phi)$.
- There is a (p.r.) function $diag(n)$ which, when applied to a number $n$ which is the g.n. of some wff, yields the g.n. of that wff’s diagonalisation [we need to make use of (p.r.) concatenation and a (p.r.) function that takes a number and returns the Goedel number of its numeral].
Goedel’s sentence

- The relation $Gdl_Z(m, n)$ is defined to hold just when $m$ is the super g.n. for a $Z$-proof of the diagonalisation of the wff with g.n. $n$.
- $Gdl_Z(m, n)$ is recursive just in case $Prf_Z$ and $diag$ are recursive.
- If $Gdl_Z$ is recursive then it can be expressed (and captured in $Q$) by $\Sigma_1$ wff $Gdl_Z$. 
Now consider \( U_Z = \forall x \neg \text{Gdl}_Z[x, x_1] \).

Now diagonalise \( U_Z \) to obtain \( G_Z \):

\[
U_Z[\neg U_Z]
\]

which is equivalent to (or we could alternatively use)

\[
\exists x_1 (x_1 = \neg U_Z \land U_Z)
\]

\( G_Z \) is \( \Pi_1 \) as it is (equivalent to) the negation of a \( \Sigma_1 \) wff.
$G_Z$ and incompleteness

$G_Z$ is true iff $G_Z$ is not derivable in $Z$.

$G_Z$ is true
iff $U_Z[\neg U_Z]$ is true
iff $\forall x \neg \text{Gd}_Z[x, \neg U_Z]$ is true
iff no $m$ is the g.n. for a $Z$-proof of the diagonalisation of the wff with g.n. $\neg U_Z$
iff $U_Z[\neg U_Z]$ is not derivable in $Z$
iff $G_Z$ is not derivable in $Z$
If $PA$ is sound, then there is a true $\Pi_1$ sentence $G$ such that
$PA \not\vDash G$ and $PA \not\vDash \neg G$, so $PA$ is negation incomplete.

First verify that the axioms of $PA$ are indeed a recursive set
(the axioms of $Q$ certainly are).

Set $G$ to be $G_{PA}$.

Then $PA \vdash G$ iff $G$ is false, so if $PA$ is sound then $G$ is true
and $PA \not\vDash G$.

But if $PA$ is sound then also $PA \not\vDash \neg G$. 

$G_z$ and incompleteness
If $T$ is a sound axiomatised theory (i.e. $Prf_T$ is recursive) whose language contains the language of basic arithmetic, then there will be a true $\Pi_1$ sentence $G_T$ such that $T \nvdash G_T$ and $T \nvdash \neg G_T$, so $T$ is negation incomplete.
Another way of looking at it

- Let $P(m, \neg \phi, n)$ hold just when $m$ numbers a derivation in $T$ of $\phi[\overline{n}]$ (i.e. of $\phi[x_1/\overline{n}]$). This is recursive.
- Now consider $P(x_1, x_2, x_2)$, this can be captured in $Q$ by the $L_{A2}$ wff $\Phi_G$.
- Now consider $\forall x \neg \Phi_G[x, \neg \forall x \neg \Phi_G[x, x_1]]$. This is a Goedel sentence.
Is this the result we want?

- If $T$ is a sound formal axiomatised theory whose language contains the language of basic arithmetic, then there will be a true sentence $G_T$ of basic arithmetic such that $T \not\models G_T$ and $T \not\models \neg G_T$, so $T$ is negation incomplete.

- The Goedel sentence we have obtained turns out to be $\Pi_1$. 
ω-completeness, ω-consistency

- A theory $T$ is ω-incomplete iff, for some open wff $\phi$, $T \vdash \phi[x/\bar{n}]$ for each natural number $n$, but $T \not\vdash \forall x \phi$.
- $\mathbb{Q}$ is ω-incomplete, consider $\forall x(x \neq sx)$.
- A theory $T$ is ω-inconsistent iff, for some open wff $\phi$, $T \vdash \phi[x/\bar{n}]$ for each natural number $n$, but $T \vdash \neg\forall x \phi$.
- Similarly for the existential quantifier.
- If $T$ is ω-inconsistent then $T$’s axioms cant all be true on an arithmetically standard interpretation.
Goedel sentence and consistency

If $PA$ is consistent, $PA \not\vdash G_{PA}$.

$PA \vdash G_{PA}$ implies some $m$ is the g.n. for a $PA$ proof of $G_{PA}$
implies some $m$ is the g.n. for a $PA$ proof of the
diagonalisation of the wff with g.n. $\neg U_{PA}$
implies $Gdl(m, \neg U_{PA})$
implies $PA \vdash Gdl_{PA}[m, \neg U_{PA}]$
implies $PA \vdash \neg \forall x_1 \neg Gdl_{PA}[x_1, \neg U_{PA}]$

But $G_{PA}$ is equivalent to $\forall x_1 \neg Gdl_{PA}[x_1, \neg U_{PA}]$ and so $PA$ is inconsistent.
Goedel sentence and $\omega$-completeness

- If $PA$ is consistent, $PA$ is $\omega$-incomplete.
  - If $PA$ is consistent then $PA \not\vdash G_{PA}$, so
    \[ PA \not\vdash \forall x \neg \text{Gdl}_{PA}[x, \overline{U_{PA}}] \]
  - but then $\neg Gdl(n, \overline{U_{PA}})$ for all $n$, so:
    \[ PA \vdash \neg \text{Gdl}_{PA}[\overline{n}, \overline{U_{PA}}] \]
    for each $\overline{n}$. So $PA$ is $\omega$-incomplete.
Goedel sentence and $\omega$-completeness

- If $PA$ is $\omega$-consistent, $PA \not\vdash \neg G_{PA}$.
  - If $PA$ is $\omega$-consistent then then $PA$ is consistent so $PA \not\vdash \neg Gd_{PA}[\overline{n}, \overline{U_{PA}}]$ for each $\overline{n}$.
  - But then $PA \not\vdash \neg G_{PA}$ implies
    \[ PA \vdash \exists x Gd_{PA}[x, \overline{U_{PA}}] \]
    and $PA$ is $\omega$-inconsistent. So $PA \not\vdash \neg G_{PA}$. 
The syntactic argument

- If $PA$ is $\omega$-consistent, then there is a $\Pi_1$ sentence $G$ such that $PA \not\vdash G$ and $PA \not\vdash \neg G$
- If $T$ is an $\omega$-consistent axiomatised theory, then there is a $\Pi_1$ sentence $G_T$ such that $T \not\vdash G_T$ and $T \not\vdash \neg G_T$
- This is the first incompleteness theorem.
Some corollaries

- If $T$ is sound and contains $\mathbb{Q}$ then $\text{theorem-of-}T$ is not recursive. Because if $\text{Prf}_T$ were recursive then it would be captured by $\text{Prf}_T$ and either $G_T$ or $\neg G_T$ would be derivable.
- If $T$ has the natural numbers as a model then $\text{theorem-of-}T$ is not recursive. Because $\text{theorem-of-}T + \mathbb{Q}$ cannot be recursive by the above.