Quantum Mechanics: the Basics II

The Dreaded Quantum, 31i2017, Jeremy Butterfield (jb56) Thanks to **Adam Caulton**

1 Recap: Inner products and Hilbert space

Vector spaces may be assigned an *inner product*, which is an assignment of a complex number to every pair of vectors $|\phi\rangle, |\psi\rangle \mapsto \langle\phi|\psi\rangle \in \mathbb{C}$.

The inner product allows us to assign a *length* (a.k.a. *norm*) to each vector; $\|\phi\| := \sqrt{\langle\phi|\phi\rangle}$. If $\|\phi\| = 1$, then $|\phi\rangle$ is said to be *normalized*. If $\langle\phi|\psi\rangle = 0$ then $|\phi\rangle$ and $|\psi\rangle$ are said to be *orthogonal*. A collection of vectors are *orthonormal* iff they are each normalized and mutually orthogonal.

It is crucial for quantum mechanics that length and orthogonality for states (represented as vectors) are definable. A *state-vector*'s length is connected to the normalization of probabilities (see Section 3) and orthogonality represents a special sort of distinctness between states (see Section 2).

The state spaces used in quantum mechanics are *Hilbert spaces*, usually denoted $\mathcal{H}$ (pronounce: "curly H"). A Hilbert space is a vector space over $\mathbb{C}$, equipped with an inner product, that is *complete in the norm* $\|\cdot\|$. (Being complete in the norm means that the limit vector of any Cauchy sequence of vectors in $\mathcal{H}$ is also in $\mathcal{H}$, where the limit is taken in terms of distance given by $\|\cdot\|$.)

2 Quantities (a.k.a. "observables")

The abstract quantities, intended to represent physical quantities like position and momentum (a.k.a. *observables*), in quantum mechanics comprise a special class of *linear operators* on $\mathcal{H}$. Linear operators are so-called because they send vectors to vectors; i.e. they preserve straight lines in $\mathcal{H}$.

Any linear operator A has *eigenvectors*, a.k.a. *eigenstates*, which are vectors in $\mathcal{H}$ that change, if anything, *only their magnitude* when acted on by A. Each eigenvector may be associated with an *eigenvalue*, which is the factor by which the length of the eigenvector changes due to having been acted on by A.

The special class of linear operators that represent physical quantities are called *Hermitian* operators. They are formally defined by $\langle\phi|A\psi\rangle = \langle A\phi|\psi\rangle$ for all $|\phi\rangle, |\psi\rangle \in \mathcal{H}$. They are characterized by having: (i) orthogonal eigenvectors; and (ii) real eigenvalues. Any Hermitian operator may be defined by specifying: (i) an *orthonormal basis* of $\mathcal{H}$; and (ii) a real number for each basis vector (this is the content of what is called the *spectral decomposition theorem*). This will be important for Section 3.

Two absolutely central quantities in quantum mechanics are position and momentum. These are defined by representing the state $|\psi\rangle$ as a complex-valued wave $\psi(x)$ over space. In one dimension, position Q and momentum P are defined as follows:

$$(Q\psi)(x) = x\psi(x); \qquad (P\psi)(x) = -i\hbar\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}. \tag{1}$$

Following the previous paragraph, we can instead define these quantities in terms of their normalized eigenvectors and eigenvalues:

- Q's eigenstates have wave representations that are *Dirac delta "functions"* $\delta(x-a)$, which, for each position a, are defined as being equal to zero for all $x \neq a$, and yet still normalized (i.e. $\int \mathrm{d}x\, \delta(x-a) = 1$). The eigenstate $\delta(x-a)$ is associated with the eigenvalue a.

- P's eigenstates have wave representations that are *plane waves* $\propto e^{ikx}$. (There is an important subtlety here that we won't pursue, that plane waves are not normalizable.) The eigenstate e^{ikx} is associated with the eigenvalue $\hbar k$, in line with de Broglie's formula $p = \frac{h}{\lambda}$.

There is another special class of linear operators important to quantum mechanics, the *unitary* operators. They are important because they preserve vector length (in fact that is how they are *defined*, in addition to their possessing an inverse). The operator $U(t)$ that takes a state at one time to a state at some defined later time is a unitary operator.

Dynamical evolution of states is governed by the *Schrödinger equation*:

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle(t) = H|\psi\rangle(t), \qquad \text{or} \qquad |\psi\rangle(t) = U(t)|\psi\rangle(0), \tag{2}$$

where H is some Hermitian operator and $U(t) = e^{-itH}$. The important thing to note here is that the equations are linear, so solutions can be superposed to generate new solutions.

3 The Born rule and the projection postulate

Finally, after all this abstract mathematics, we link up with the physical world! In the "minimal" formulation of quantum mechanics, the following rules are the *only* interpretative connection to the physical world, and they concern only the performing of *experiments* (this is the source of their generality *and* problematic nature).

Let A be any Hermitian operator. Then let $\{|a_i\rangle\}$ be an orthonormal set of its eigenstates, where $|a_i\rangle$ is associated with the (real numbered) eigenvalue a_i (so $A|a_i\rangle = a_i|a_i\rangle$). And let the state of our quantum system be $|\psi\rangle$. Then the **Born rule** states:

> *The result of the measurement of the quantity A on the system in state $|\psi\rangle$ yields the eigenvalue a_i with probability $|\langle a_i|\psi\rangle|^2$.*

Since the eigenvalues a_i exhaust the possible values of A, and any outcome is distinct from any other, we expect the probabilities given by the Born rule to *normalize*, i.e. sum to 1:

$$\sum_i |\langle a_i|\psi\rangle|^2 = 1 \tag{3}$$

But, due to a generalization of Pythagoras' theorem, $\sum_i |\langle a_i|\psi\rangle|^2 = \|\psi\|^2$. We therefore demand that physical states be represented by *normalized* vectors: $\|\psi\| = 1$.

Now the **Projection Postulate** states:

> *Immediately after a measurement of the quantity A, the system's state is the eigenstate $|a_k\rangle$ of A corresponding to the measured eigenvalue a_k.*

Thus the *act of measurement* forces the system's state to "jump" to one of the eigenstates determined by the quantity being measured.

4 Incompatible quantities and the uncertainty principle

We are finally ready to present one of the most important prediction of quantum mechanics: *the uncertainty principle*. I will state it here as uncontroversially as possible.

Given a particular state $|\psi\rangle$, any quantity A may be associated with a certain *statistics* for the outcomes of its measurements. That is, quantum mechanics predicts determinate probabilities, given by the Born rule, for each of the possible outcomes—each of the eigenvalues of A—which result from a measurement of A on the system. These probabilities can be confirmed/disconfirmed (as for chancy systems generally) only by collecting statistics for a large number of repeated trials.

The mathematics behind the uncertainty principle is that two different quantities, A and B say, may be associated with *different* eigenstates. But then it follows from the projection postulate that we cannot perform a measurement of *A and B simultaneously*, for there is no unique class of eigenstates for the state to then "jump" to immediately afterwards. We say that A and B are *incompatible*, or *incommensurable*. Examples of incompatible pairs are: position Q vs. momentum P; and spin in the z-direction, σ_z, vs. spin in the x-direction, σ_x.

We could measure B very soon after measuring A, but the projection postulate tells us that the probabilities of outcomes for the second measurement (of B) will be determined not by the state the system was in *before* the first measurement (of A), but rather by its state *after* the first measurement, which must be an eigenstate of A. Similarly, consider a measurement of A performed very soon after a measurement of B. If A and B are incompatible, then, typically, different statistics for A and B will be obtained in each case.

This difference is quantified by the *commutator* $[A, B] := AB - BA$. In particular, given any state $|\psi\rangle$, we may calculate the *expectation value of* the commutator $[A, B]$, written $\langle[A, B]\rangle$, for $|\psi\rangle$. A and B are incompatible iff $[A, B] \neq 0$. We may say that A and B are *maximally incompatible* iff there is no state such that $\langle[A, B]\rangle = 0$. The pairs position and momentum, and spin-z and spin-x, are maximally incompatible.

The uncertainty principle says that the statistics for A and B associated with any state are constrained by the value of $\langle[A, B]\rangle$ for that state. To articulate this more fully, we define a quantity ΔA, colloquially known as the "uncertainty" in A, but really it is the *standard deviation* $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2} \equiv \sqrt{\langle(A - \langle A\rangle)^2\rangle}$ in values obtained from measurements of A. We similarly define ΔB. Then uncertainty principle is

$$\Delta A \Delta B \geqslant |\langle[A, B]\rangle|. \tag{4}$$

In particular, for position and momentum we have

$$\Delta Q \Delta P \geqslant \hbar. \tag{5}$$

This means that *we cannot achieve zero spread in the statistics for both Q- and P- measurements, no matter what the state.* If we identify "sharp" statistics (i.e. zero spread) in Q as particle-like behaviour and "sharp" statistics in P as wave-like behaviour (remember de Broglie's hypothesis), then the uncertainty principle implies a fundamental incompatibility between the particle- and wave-pictures of a quantum system.

5 Summary: 5 principles

Cf. Albert (1992, 30-36).

1. *States.* The state space is a Hilbert space, which is a special kind of vector space, and physical states are represented by unit vectors $\equiv$ normalised wave-functions.

2. *Quantities.* Physical quantities are represented by Hermitian operators, which are a special kind of linear operator. In particular, as operators on wave-functions, $Q : \psi(x) \mapsto x\psi(x)$ and $P : \psi(x) \mapsto -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)$. These quantities are *incompatible*, in that they share no eigenstates, and so may not be measured simultaneously.

3. *Dynamics: the Schrödinger equation.* The dynamical evolution of the state $|\psi\rangle(t)$ over time is given by the equation $i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle(t) = H|\psi\rangle(t)$ for some Hermitian operator H.

4. *Measurement 1: the Born rule.* The result of any measurement of the quantity A on the system in state $|\psi\rangle$ yields the eigenvalue a_i with probability $|\langle a_i|\psi\rangle|^2$.

5. *Measurement 2: the projection postulate.* Immediately after a measurement of the quantity A, the system's state is one of the eigenstate $|a_k\rangle$ of A, corresponding to the measured eigenvalue a_k.

6 Further Reading

- $*$ Albert, David Z. (1992), *Quantum Mechanics and Experience* (Cambridge, MA and London: Harvard UP), pp. 1-47.

- $*$ Cushing, J. (1998), *Philosophical Concepts in Physics* (Cambridge: CUP 1998), Ch. 19.

- Hughes, R. I. G. (1989), *The Structure and Interpretation of Quantum Mechanics* (Cambridge, MA and London: Harvard UP), Chs. 1-5.

- Redhead, Michael (1987), *Incompleteness, Nonlocality and Realism* (Oxford: Clarendon), §§1.1-1.5 (pp. 1-30).