Philosophy of Mathematics
Introduction

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Talk outline

Introduction

Benacerraf's Dilemma

Mathematical logic

Questions in the philosophy of mathematics

Ontological	Do mathematical objects like numbers, sets and points exist?
Metaphysical	What is the nature of mathematical objects?
Semantic	How do we refer to mathematical objects? Are mathematical sentences truth-apt?
Epistemological	How can we know about mathematical objects?
Practical	How is it that mathematics is essential to science?

Why have a philosophy of mathematics?

- What's special about the philosophy of *mathematics*?
    1. Mathematics is a body of universal and necessary truths.
    2. It is *a priori*.
    3. It seems immune to inductive confirmation.
    4. Its methodology is *apodeictic*: it proceeds by proof.
    5. It seems to be separate from empirical science, yet indispensable to science.
    6. Its subject matter is *infinitary*.

This course

- This lecture will introduce some of the issues in the philosophy of maths.
- We will also cover some of the logic and mathematics needed.
- In the remainder, we'll discuss 4 of the major contemporary schools in the philosophy of mathematics:
  - Logicism: mathematics can be reduced to logic.
  - Structuralism: mathematics is the science of structure.
  - Nominalism: mathematics is all false though useful.
  - Intuitionism: mathematics is constructed by humans.
- The major school we will not be discussing is *formalism*: that is on the Mathematical Logic paper and will be covered in the Gödel lectures.

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Benacerraf's Dilemma

- Paul Benacerraf, in 'Mathematical Truth' (1973), pointed out that the most obvious answers to the questions 'What is a human?' and 'What is mathematics?' seem to make mathematical knowledge impossible.
  - (A) $4 > 2$
  - (B) Oxford is larger than Cambridge
- In (B), 'Oxford' and 'Cambridge' are singular terms referring to objects.
- (A) and (B) look formally similar.
- So perhaps '4' and '2' are singular terms referring to objects. But what could the objects be?
- Not concrete particulars, certainly.
- They must be are abstract, causally isolated from the spatiotemporal world.

Benacerraf's Dilemma

- In post-Gettier epistemology, we no longer think that knowledge is justified true belief: there must be some extra ingredient.
- The usual strategy is to postulate some *connection* between the knower and the fact known.
- This is to rule out the case where we know by luck.
- But how can we be so connected to mathematical facts?
- They are abstract and so not the sorts of things we concrete creatures can interact with.

Responses

- The metaphysics and the epistemology therefore seem in tension.
- Philosophers of mathematics can either accept the easy metaphysical story or the easy epistemological story.
- But whichever they take will leave significant work on the other horn.
- The philosophers of mathematics that we'll be discussing can be helpfully mapped onto Benacerraf's dilemma.

Responses

- Frege, and neo-Fregeans, accept that numbers are abstract objects. Epistemologically, they try to reduce mathematical truths to logical ones.
- Structuralists also hold that numbers are abstract objects, but of a particular kind: we can discern them by seeing patterns in concrete objects.
- Kant, and the intuitionists, deny that numbers are objects: they are mental projections from humans.
- Nominalists deny that numbers exist at all. As a result, mathematical sentences are false, and so cannot be known.

Talk outline

Robinson Arithmetic

- Arithmetic will be the central part of mathematics that we consider.
- The main arithmetic theory we'll consider is Peano Arithmetic.
- To introduce that, let's first consider Q, which is characterised by the following axioms:
  1. $\forall x(0 \neq Sx)$
  2. $\forall x \forall y(Sx = Sy \rightarrow x = y)$
  3. $\forall x(x + 0 = x)$
  4. $\forall x \forall y(x + Sy = S(x + y))$
  5. $\forall x(x \times 0 = 0)$
  6. $\forall x \forall y(x \times Sy = (x \times y) + x)$
- This is very weak arithmetic: it can't even prove that addition is symmetric.

Peano Arithmetic

- Peano Arithmetic is the theory that adds an *induction schema* to Q:
  - I $(\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(Sx))) \rightarrow \forall x \phi(x)$
- Here, $\phi(x)$ is an open wff that has x free.
- The familiar basic truths about the successor function, addition, multiplication and ordering are all provable in PA.
- PA has infinitely many axioms: all the instances of I.

Motivating second-order logic

- Consider the argument:

  Kanye is a rapper, Kanye will be president in 2024; so someone is a rapper and will be president in 2024.

- Rk, Pk; *so* $\exists x(Rx \wedge Px)$
- Now consider:

  Kanye is a philosopher, Plato is a philosopher; so there is something that Kanye and Plato both are.

- There is no good way to formalise it in first-order logic.
- In the latter, we are quantifying over *properties* rather than *objects*, and this cannot be expressed at first-order.

Second-order logic

- Second-order logic allows for this quantification over properties.
- It is an extension of first-order logic:
  - To the language, we add predicate variables of each degree: X_1, X_2, ..., Y_1, Y_2, ...
  - To the grammar, we add clauses allowing predicate variables, as well as predicate constants, to appear in formulas, and to follow $\exists$ and $\forall$.
  - Second-order variables range over the powerset of the domain, e.g. $\forall X \phi$ is true in an interpretation just if ϕ is true of every subset of the domain; $\exists X \phi$ is true in an interpretation just if ϕ is true of some subset of the domain.
- Our Kanye argument can now be formalised: Pk, Pp; *so* $\exists X(Xk \wedge Xp)$.

Strength of second-order logic

- Second-order logic is expressively very strong.
- There is no consistent proof theory with respect to which second-order logic is complete: it is not *axiomatisable*.
- Its strength has led some to believe that it is not really logic. Quine famously called it ‘set theory in sheep’s clothing’.
- The thought is that, by quantifying over all subsets of the domain, we are no longer doing logic but set theory. We’ll return to this question.

Second-order Peano Arithmetic

- Given the expressive power of second-order logic, we can express the principle of mathematical induction:
  - I $(\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(Sx))) \rightarrow \forall x \phi(x)$
  - I′ $\forall X((X(0) \wedge \forall x(X(x) \rightarrow X(S(x)))) \rightarrow \forall x X(x))$
- Second-order Peano Arithmetic is the theory obtained by replacing the induction schema, I, with the induction axiom, I′.
- I′ is strictly stronger than I.
- Why? The second-order variables in I′ range over the set of all subsets of the natural numbers, whereas I has only as many instances as there are natural numbers.
- The former is larger than the latter, by Cantor's theorem: for any set S, the powerset of S has strictly greater cardinality than S. $2^{\aleph_0} > \aleph_0$.

Non-standard models

- Why is this increased strength important?
- First-order Peano Arithmetic has *non-standard* models.
- By the Löwenheim-Skolem theorem, if a first-order theory has a model, then it has a countable model (the size of the natural numbers).
- Say we want a theory of the real numbers. The intended interpretation of our theory will have an uncountable domain.
- But there will a non-standard model with a domain that is merely countable.

Categoricity

- It is here that some people think second-order theories have an advantage.
- Second-order theories are *categorical*: all of their models are isomorphic.
- That means, roughly, that all models of a second-order theory are the same size: there is a one-to-one correspondence between their members.
- So the problem of non-standard models that arises for first-order theories is avoided by second-order theories.
- There can still be *non-standardness* (e.g. Putnam's permutations) but not in the size of domain.

Completeness and consistency

- A theory T is *(negation) complete* iff, for every sentence ϕ, $T \vdash \phi$ or $T \vdash \neg\phi$.
- NB This is not to be confused with the completeness of a *logic*, which means that every logical consequence is provable. We are here discussing the completeness of *theories*.
- A theory T is *consistent* iff there is no sentence ϕ such that $T \vdash \phi$ and $T \vdash \neg\phi$.
- This is *syntactic* or *proof-theoretic* consistency. There is a corresponding notion of semantic consistency.

Gödel's incompletenss theorems

- This allows us to state two well-known results:

  First incompleteness theorem Any consistent theory T that is at least as strong as Q is incomplete.

  Second incompleteness theorem Any consistent theory T that is at least as strong as Q cannot prove its own consistency.
- Remember, Q is *very* weak. So any remotely interesting theory will be incomplete.
- This means that Peano Arithmetic is incomplete, whether first- or second-order.
- These results cause problems for various views, most famously *formalism*.

Euclidean geometry

- The standard geometry we will consider is Euclidean geometry:
  1. Given any two points P and Q, exactly one line can be drawn which passes through P and Q.
  2. Any line segment can be indefinitely extended.
  3. A circle can be drawn with any centre and any radius.
  4. All right angles are congruent to each other.
  5. If a line l intersects two distinct lines m and n such that the sum of the interior angles a and b is less than 180°, then m and n will intersect at some point.

Hyperbolic geometry

- Axioms 1–4 are really just abstractions from what we can construct with a rules, compass and protractor, but Axiom 5 is different.
- We may have to travel an *extremely* long distance before m and n intersect, and so may not be able to draw the relevant lines.
- For this reason, mathematicians in the 19th century consider alternatives to Axiom 5.
- In the extreme case:
  - $\neg$ 5 There exists a line l and point P not on l such that at least two distinct lines parallel to l pass through P.
- This is the *hyperbolic axiom*. The geometry that replaces 5 with the hyperbolic axiom is hyperbolic geometry. It is consistent if Euclidean geometry is.