

# Philosophy of Mathematics

## The roots of logicism

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## Completeness and consistency

- ▶ A theory  $T$  is (*negation*) *complete* iff, for every sentence  $\phi$ ,  $T \vdash \phi$  or  $T \vdash \neg\phi$ .
- ▶ NB This is not to be confused with the completeness of a *logic*, which means that every logical consequence is provable. We are here discussing the completeness of *theories*.
- ▶ A theory  $T$  is *consistent* iff there is no sentence  $\phi$  such that  $T \vdash \phi$  and  $T \vdash \neg\phi$ .
- ▶ This is *syntactic* or *proof-theoretic* consistency. There is a corresponding notion of semantic consistency.

## Gödel's incompleteness theorems

- ▶ This allows us to state two well-known results:
  - First incompleteness theorem** Any consistent theory  $T$  that is at least as strong as  $Q$  is incomplete.
  - Second incompleteness theorem** Any consistent theory  $T$  that is at least as strong as  $Q$  cannot prove its own consistency.
- ▶ Remember,  $Q$  is *very* weak. So any remotely interesting theory will be incomplete.
- ▶ This means that Peano Arithmetic is incomplete, whether first- or second-order.
- ▶ These results cause problems for various views, most famously *formalism*.

# Talk outline

The analytic/ synthetic distinction

Kant: arithmetic as synthetic

Frege: arithmetic as analytic

Conclusion

## Frege's aims

- ▶ Frege's central aim in the *Grundlagen* was to establish the truth of *logicism*: arithmetic truths are logical truths.
- ▶ A subsidiary aim was to knock down competing philosophies of mathematics, especially Kant's. For Kant, mathematical truths are synthetic (hence non-logical).
- ▶ But he agreed with Kant on some points:

*I have no wish to incur the reproach of picking petty quarrels with a genius to whom we must all look up with grateful awe; I feel bound, therefore, to call attention also to the extent of my agreement with him, which far exceeds my disagreement. ... In calling the truths of geometry synthetic and a priori, he revealed their true nature. ... His point was that there are such things as synthetic judgements a priori, whether they are to be found in geometry only, or in arithmetic as well, is of less importance.*  
(§89)

## The *a priori*/ *a posteriori* distinction

- ▶ An epistemological distinction.
- ▶ Kant applies it to *cognition*. I will apply it to *knowledge*.
- ▶ A truth is known *a priori* if it is known 'absolutely independently of all experience and even of all impressions of the senses' (B2–3).
- ▶ '[E]xperience teaches us ... that something is constituted thus and so, but not that it could not be otherwise' (B3).
- ▶ '[E]xperience never gives its judgements true or strict universality' (B3).
- ▶ *A priori* knowability entails necessity and universality/ generality.

# The analytic/ synthetic distinction

- ▶ A semantic distinction.
- ▶ Now, a standard definition is e.g.
  - A sentence is analytic iff it is true by definition.
  - A sentence is analytic iff it is a tautology, or reducible to one by substitution.
  - A sentence is analytic iff it is true in virtue of the meanings of the words.
- ▶ Such definitions face well-known problems from Quine.
- ▶ And these are not the sorts of definitions offered by Kant.

## The analytic/ synthetic distinction

- ▶ Kant applies the distinction to *judgements*. I will apply it to *sentences*.
- ▶ A sentence in subject-predicate form is *analytic* 'if the predicate *B* belongs to the subject *A* as something that is (covertly) contained in the concept *A*' (A6/B10).
- ▶ Otherwise, it is *synthetic*.
- ▶ Analytic sentences add 'nothing through the predicate to the concept of the subject, but merely [break] it up into those constituent concepts that have been thought in it, although confusedly' (A7/B11).

## Conceptual containment

- ▶ This definition relies on the obscure *conceptual containment* metaphor.
- ▶ Frege criticises the metaphor in relation to mathematics:

*Kant obviously – as a result, no doubt, of defining them too narrowly – underestimated the value of analytic judgements ... [Mathematical concepts] are contained in their definitions, but as plants are contained in their seeds, not as beams are contained in a house. Often we need several definitions for the proof of some proposition, which consequently is not contained in any of them alone. (Grundlagen §88)*
- ▶ And it only applies to sentences in subject-predicate form.

## The analytic as explicative

- ▶ Kant offered another definition of the analytic that avoids these issues.
- ▶ He (A7/B11) characterised analyticity as being *explicative*, rather than *ampliative*.
- ▶ If we understand an analytic sentence, we learn nothing new when we come to know its truth.
- ▶ In this sense, analytic sentences cannot *amplify* our knowledge: they are merely *explicative*.

## Law of contradiction

- ▶ Kant gives us two tests for analyticity in this sense.
- ▶ First, analytic sentences 'can always be adequately known in accordance with the principle of contradiction' (A150–153/B189–192).
- ▶ If the negation of a sentence  $S$  is a contradiction, then  $S$  is analytic.
- ▶ By this test, all *logical* truths are obviously analytic.

## Independence of objects

- ▶ Second, the truth of a synthetic sentence requires something else for its truth.
- ▶ Because they are ampliative, synthetic sentences depend on the existence of at least one object.
- ▶ The truth of an analytic sentence is independent of all objects whatsoever.

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## Law of contradiction

- ▶ By these tests, are arithmetic truths analytic?
- ▶ Let's start with the law of contradiction.
- ▶ ' $7+5 \neq 13$ ' is an arithmetic truth.
- ▶ Is its negation, ' $7+5=13$ ', a contradiction?

## Law of contradiction

- ▶ Kant writes:

*a synthetic sentence can indeed be discerned in accordance with the principle of contradiction, this can only be if another synthetic sentence be presupposed; ... it can never be so discerned in and by itself (B14)*

- ▶ A statement of identity (as opposed to non-identity) can only be a contradiction in the presence of other assumptions.
- ▶ So some arithmetic truths only have contradictory negations in the presence of other assumptions, namely *axioms*.

# Axioms

- ▶ Is the negation of an *axiom* a contradiction?
- ▶ Kant thought not: that would make it a logical truth, so we wouldn't need it as an axiom.
- ▶ Axioms should be deniable without contradiction.
- ▶ The law of contradiction gives us reason to think that axioms are synthetic.
- ▶ Any mathematical truth that depends essentially on an axiom is likewise synthetic.

# Independence

- ▶ How about the other test: independence from objects?
- ▶ Taken at face-value, the axioms of arithmetic appear to require the existence of objects, e.g. a referent for '0'.
- ▶ We could paraphrase these away, or attempt to reduce them to logic.
- ▶ But, as we'll see, paraphrase projects still need objects, e.g. space-time points.
- ▶ And logicism needs second-order logic, which has amongst its logical truths sentences like:

$$\exists X \forall x (Xx \leftrightarrow x \neq x)$$

$$\exists X \forall x (Xx \leftrightarrow x = x)$$

## Generality

- ▶ Is this too stringent a condition? After all, first-order logic (classically understood) has  $\exists x x = x$  as a logical truth.
- ▶ Reply: move to an *inclusive* logic.
- ▶ Kant holds that the non-ampliativity of logic is essential. This is what gives logic its distinctive *generality*.
- ▶ Logic should not be in the business, even partially, of answering the question, 'What is there?'.
- ▶ Thus Kant: I 'admit, as every reasonable person must, that all existential propositions are synthetic' (A598=B626).

## Modern mathematics

- ▶ Does the development of mathematics since Kant support his claim that mathematics is synthetic?
- ▶ Hilbert's completion of the axiomatisation of Euclidean geometry, and Peano's of arithmetic, certainly appear synthetic in Kant's sense.
- ▶ Other areas, however, seem to pull against Kant. E.g. group theory doesn't seem to consist of *truths* in the same way as arithmetic and geometry.
- ▶ Many hold an *implicationist* view of group theory: that theorem  $\phi$  follows from axioms  $\Gamma$  should be understood as 'if  $\Gamma$ , then  $\phi$ ', rather than the categorical assertion of  $\phi$ .

## The axiomatic method

- ▶ So, for Kant, the analytic sentences are the explicative (non-ampliative) statements.
- ▶ (Most) mathematical sentences are synthetic by this definition: they rely on axioms, whose negations are not contradictions, and require the existence of objects for their truth.
- ▶ This reading is somewhat anachronistic. Kant did not share the modern notion of an axiom:

*Certainly, arithmetic has no axioms, since its object is actually not any quantum, that is any quantitative object of intuition, but rather quantity as such, that is, it considers the concept of a thing in general by means of quantitative determination.*

*(Correspondence, p. 284)*

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## The law of contradiction

- ▶ Frege thought that consideration of the law of contradiction favoured his position.

*For purposes of conceptual thought we can always assume the contrary of some or other of the geometrical axioms, without involving ourselves in any self-contradictions ... Can the same be said of the fundamental propositions of the science of number? Here, we have only to try denying any one of them, and complete confusion ensues. Even to think at all is no longer possible. (§14)*

# The law of contradiction

- ▶ We can imagine worlds with radically different geometries, e.g. we now know of hyperbolic geometries.
- ▶ But can we really imagine a world in which  $2+2=5$ ?
- ▶ Doesn't it pull against the thought (which Kant accepts) that mathematics is necessary?

## The nature of arithmetic

- ▶ For these reasons, he thinks arithmetic is much more general than geometry.

*The wildest visions of delirium, the boldest inventions of legend and poetry, where animals speak and stars stand still, where men are turned to stone and trees turned into men, where the drowning haul themselves up out of swamps by their topknots – all these remain, so long as they remain intuitable, still subject to the axioms of geometry. ... The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. (§14)*

## The nature of logic

- ▶ Frege is beginning to suggest that arithmetic is so general that it is in fact a *part* of logic: the most general science.
- ▶ If we could show that arithmetic is a part of logic, we could explain its applicability to 'everything thinkable'.

*The truths of arithmetic would then be related to those of logic in much the same way as the theorems of geometry to the axioms. Each one would contain concentrated within it a whole series of deductions for future use and the use of it would be that we need no longer make the deductions one by one, but can express simultaneously the result of the whole series. ... [This] will suffice to put an end to the widespread contempt for analytic judgements and to the legend of the sterility of logic. (§17)*

## The nature of logic

- ▶ This role for logic is only thinkable given Frege's innovations in the *Begriffsschrift*.
- ▶ In traditional Aristotelian syllogistic, there is a sense in which we are merely 'taking out of the box again what we have just put into it' (§88).
- ▶ Aristotle introduced the crucial idea of *variables* to express generality.
- ▶ We can express *syllogisms* such as

**Barbara** All *F*s are *G*s, All *H*s are *F*s; so All *H*s are *G*s

**Darii** All *F*s are *G*s, Some *H*s are *F*s; so Some *H*s are *G*s

- ▶ This was still very much Kant's notion of logic. It resembles *conceptual containment*.

# Objects

- ▶ Frege's logic allows for conclusions that 'extend our knowledge' (§88).
- ▶ One hallmark of ampliative, synthetic reasoning, we have seen, is dependence on objects.
- ▶ It is the use of *variables* that allows this.
- ▶ We can, to use Dummett's example, extract the distinct concepts of murder and suicide from the sentence 'Cassius killed Caesar': 'x killed y', 'x killed x'.

# Objects

- ▶ Aristotle's (and Kant's) logic was essentially the *monadic fragment* of Frege's.
- ▶ The power of Frege's insight is captured in the metatheory: polyadic quantified logic is not decidable.
- ▶ It may *appear*, when we work in monadic first-order logic, that we are reasoning about concepts, e.g. All *As* are *Bs*.
- ▶ But this is because we have *suppressed* the variable:  
$$\forall x(Ax \rightarrow Bx)$$
- ▶ In polyadic logic, this suppression is impossible.

## The analytic/ synthetic distinction

- ▶ We might think that the use of variables here reveals a certain dependence on *objects*.
- ▶ But Frege wants to show  
*how pure thought (regardless of any content given through the senses or even given a priori through an intuition) is able, all by itself, to produce from the content which arises from its own nature judgements which at first glance seem to be possible only on the grounds of some intuition. (Begriffsschrift, §23)*
- ▶ Frege needs to show that logic alone can introduce us to objects.
- ▶ This is the major task of his logicism.

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- ▶ Frege wanted to establish *contra* Kant that arithmetic is analytic.
- ▶ Kant had two tests for analyticity: the law of contradiction and dependence on objects.
- ▶ By the first test, Frege thought that arithmetic was analytic: it is *logic* so of course its negation is contradictory.
- ▶ Now he has to undermine the second test and show how logic alone can bring about a commitment to objects.
- ▶ That's our topic next week.