Philosophy of Mathematics
Frege’s logicism

Owen Griffiths
oeg21@cam.ac.uk

Churchill and Newnham, Cambridge

18/10/16
Last week

- Kant saw mathematics as a body of synthetic, *a priori* truths.
- Two tests: contradiction, dependence on objects
- Frege agreed about geometry but disagreed about arithmetic.
- Arithmetic truths are analytic by the first test.
- But how can logic be committed to objects?
Talk outline

Principles guiding the Grundlagen

Hume’s Principle

Frege’s Theorem
Three principles

In the enquiry that follows, I have kept to three fundamental principles:

- always to separate the psychological from the logical, the subjective from the objective;
- never to ask for the meaning of a word in isolation, but only in the context of a proposition;
- never to lose sight of the distinction between concept and object.
Anti-psychologism

- Psychologism: mathematical entities are mental
- Two arguments against this view:
  - Ideas are private, whereas numbers don’t seem to be.
  - Some numbers have never been thought of.
- The *Grundlagen* is widely considered to have demolished psychologism.
Concept and object

- **Object**: entity referred to by a *singular term*
- **Concept**: entity referred to by a *predicate*
- Frege draws a sharp distinction between object and concept.
- The distinction between sense and reference was not clear at the time of writing *Grundlagen*.
Concepts

- From ‘Frege is a logician’, we can form first-level predicate ‘\(x\) is a logician’
- First-level predicates pick out first-level concepts.
- From ‘\(\forall x \; x \text{ is a logician}\)’, we can form second-level predicate ‘\(\forall x Xx\)’
- Second-level predicates pick out second-level concepts.
- Concepts have *extensions*.
- *Objects* fall under first-level concepts.
- *First-level concepts* fall under second-level concepts.
Objects

- An object is anything that can be united with a subject to form a judgement.
- To be a singular term is to function linguistically in the correct manner.
- Frege’s criterion for objecthood is *semantic*.
- This notion of objecthood is captured by the Context Principle.
The Context Principle

▶ The context principle instructs us ‘never to ask for the meaning of a word in isolation but only in the context of a proposition’.

▶ Michael Dummett argues that the context principle marks a fundamental shift in philosophy: the linguistic turn (compare: Kant’s Copernican turn).

    ... it is only in the context of a sentence that a word has meaning: the investigation therefore takes the form of asking how we can fix the senses of sentences containing words for numbers. (Dummett 1993: 5)
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Implicit definition

- A major use of the Context Principle is to justify *implicit definition*.
- *Explicit* definitions define expressions in terms of previously understood expressions: ‘vixen’ means female fox.
- *Implicit* definition define expressions functionally: Jack the Ripper committed the Whitechapel murders.
- Explicit definitions *mention* the definiendum.
- Implicit definitions *use* it.
- Frege offered an implicit definition of number.
Counting

- Suppose I want to know whether the number of knives is the same as the number of forks.
- I could count the knives, then the forks, then compare.
- In Fregean terms, I would be assigning numbers of objects to the first-level concepts *knife on the table* and *fork on the table.*
One-one correspondence

- Frege (§63) notes that there is another way, suggested by Hume:
  
  *When two numbers are so combined as that the one has always an unit answering to every unit of the other, we pronounce them as equal*  
  (*Treatise, bk 1, part iii, §1*)

- We could pair the knives and forks off.
- This establishes a *one-one correspondence*.
- So the number of knives is the same as the number of forks.
Counting

- Frege thought that *all* counting was like this.
- When we count objects, we put them into one-one correspondence with an initial segment of the natural numbers.
- Say that there is a one-one correspondence between the knives and the numbers 1 to $n$.
- Say that there is a one-one correspondence between the forks and the numbers 1 to $m$.
- Then there is a one-one correspondence between the knives and forks if, and only if, $n$ is $m$. 
One-one correspondence can be logically defined.

Relation $R$ is a one-one correspondence between the $F$s and the $G$s iff:

1. No object bears $R$ to more than one object, and
2. no object is borne $R$ by more than one object, and
3. every $F$ bears $R$ to some $G$, and
4. every $G$ is borne $R$ by some $F$.

In symbols:

\[
\forall x (Fx \rightarrow \exists! y (Gy \land Rxy)) \land \\
\forall y (Gy \rightarrow \exists! x (Fx \land Rxy))
\]
Hume’s Principle

Let’s say there is such a one-one correspondence:
\[
\exists X (\forall x (F x \rightarrow \exists! y (G y \land X xy)) \land \\
\forall y (G y \rightarrow \exists! x (F x \land X xy)))
\]

And let’s abbreviate this long expression as:
\[ F \sim G \]

We are now in a position to define numerical identity:
\[ HP \quad Nx F x = Nx G x \iff F \sim G \]
Abstraction Principles

- HP is an *abstraction principle*: it purports to give us semantic and epistemological access to abstract objects.
- Abstraction principles have the following shape:
  \[ \Sigma a = \Sigma b \leftrightarrow Eab \]
- Here, ‘\(\Sigma\)’ is a *term-forming* operator.
- It denotes a function from items in the range of the first-order variables to objects.
- ‘\(E\)’ expresses an equivalence relation over the items in the range of the first-order variables.
Abstraction Principles

- The equivalence relation $E$ is already understood.
- The kinds of objects in its range are uncontroversial.
- The truth-conditions of instances of ‘$Eab$’ are unproblematic.
- The principle tells us that the truth-conditions of identity statements involving ‘$\Sigma$’ are coincident with these unproblematic ones.
- Overall, we can exploit the entities on the RHS to explain our knowledge of, and reference to, the entities on the LHS.
Hume’s Principle

- In the case of HP, we can exploit our access to one-one correspondence in order to access numbers.
- The first-order variables in HP range over everyday objects.
- By coming to know the truth-conditions of instances of the RHS of HP, we can come to know those of the LHS.
- And if the LHS instances are true identity statements, then ‘NxFx’ must be a genuine singular term.
- For Frege, to be an object is to be referred to by a singular term.
Talk outline

Principles guiding the Grundlagen

Hume’s Principle

Frege’s Theorem
The significance of HP

- *Frege Arithmetic* is the theory built on second-order logic with HP as its only axiom.
- *Frege’s Theorem* is the remarkable result that we can derive all of the second-order Peano axioms as theorems of Frege Arithmetic.
- Frege does not prove Frege’s Theorem in *Grundlagen*.
- The proof was conjectured by Crispin Wright and proved by, amongst others, George Boolos (see his ‘On the proof of Frege’s Theorem’, 1988).
To prove Frege’s Theorem, we need to show that all of the axioms of $PA^2$ are theorems of Frege Arithmetic.

Let’s sketch how this result might be proved.

We also need to define the successor function. Frege instead defines a predecessor function:

$$ Pmn =_{df} \exists F \exists y (n = NxFx \land Fy \land m = Nx(Fx \land x \neq y)) $$

In words: $m$ is the predecessor of $n$ just if $n$ is the number of $F$s, for some $F$, and $m$ is the number of $F$s excluding one object.
Proof sketch

- Consider the following axiom of $PA^2$:
  \[ Ax \ \forall x \forall y (Sx = Sy \rightarrow x = y) \]

- In terms of predecession, this can be expressed as:
  \[ Ax' \ \forall x \forall y \forall z (Pxz \land Pyz) \rightarrow x = y \]

- To prove that $Ax'$ is a theorem of Frege Arithmetic, we need the following Equinumerosity Lemma:
  \[ Eq \ (F \sim G \land Fa \land Gb) \rightarrow F^{-a} \sim G^{-b} \]

- Here, $F^{-a}$ is the $Fs$ apart from $a$. 

Proof sketch

- Assume $Pac$ and $Pbc$.
- By the definition of $Pxy$,

\[
Pmn =_{df} \exists F \exists y(n = NxFx \land Fy \land m = Nx(Fx \land x \neq y))
\]

we know that there will be concepts $F$, $G$ and objects $d$ and $e$ such that:

- $c = NxFx \land Fd \land a = NxF^{-d}x$
- $c = NxGx \land Ge \land b = NxG^{-e}x$

- If $c = NxFx$ and $c = NxGx$, then $NxFx = NxGx$
- By HP, $F \sim G$
- By Eq, $F^{-d} \sim G^{-e}$. $[(F \sim G \land Fd \land Ge) \rightarrow F^{-d} \sim G^{-e}]$
- By HP, $NxF^{-d}x = NxG^{-e}x$.
- So $a = b$
- So $(Pac \land Pbc) \rightarrow a = b$
- So $\forall x \forall y \forall z(Pxz \land Pyz) \rightarrow x = y)$
Frege rejected HP as a definition of number.

Given the context principle, we have succeeded in settling the meaning of an expression just if we have settled the meaning of every sentence in which that expression features.

But HP does not settle the meaning of all sentences involving number terms. It only settles those of the form ‘\(NxFx = NxGx\)’.
Frege sums up the worry: 

*we can never – to take a crude example – decide by means of our definitions whether any concept has the number Julius Caesar belonging to it, or whether that same familiar conqueror of Gaul is a number or not.* (§56)

HP allows us to settle the truth value of ‘$NxFx = NxGx$’: it’s true just if the $F$s and $G$s can be put in one-one correspondence).

But it is silent as to the truth value of ‘$NxFx = Julius Caesar$’.

This problem is now known as the Julius Caesar problem.
Explicit definition

In light of the Julius Caesar problem, Frege opts for an *explicit* definition of number:

*My definition is therefore as follows:*

*the Number which belongs to the concept $F$ is the extension of the concept “equal to the concept $F”*(§68)

By this definition, numbers are *extensions of second-level concepts.*

The extension of a second-level concept is all of the first-level concepts that fall under that second-level concept.
The extension of concept $F$ is $\{x : Fx\}'$

$$NxFx =_d f \{X : X \sim F\}$$

‘The number of members of The Beatles’ is a term picking out the set of all concepts equinumerous with the first-level concept *member of the Beatles*.

One such first-level concept is *prime numbers less than or equal to 7*.

There is a second-level concept under which only these and other 4-membered first-level concepts fall.

This second-level concept has an extension, and its extension is identical to the number of members of The Beatles.
Extensions

- Does the explicit definition solve the Julius Caesar problem?
- ‘I assume that it is known what the extension of a concept is’ (§68, n.1).
- The explicit definition enjoys all of the technical benefits of HP, since
  \[ NxFx =_{df} \{ X : X \sim F \} \]
  straightforwardly entails:
  \[ NxFx = NxFx \iff F \sim G \]
- Frege came to spell out the details in *Grundgesetze der Arithmetik* (*Basic Laws of Arithmetic*, first volume published in 1893, second in 1903).
- Here, the problems with his definition in terms of extensions became apparent.