Philosophy of Mathematics
Frege's logicism

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Last week

- Kant saw mathematics as a body of synthetic, *a priori* truths.
- Two tests: contradiction, dependence on objects
- Frege agreed about geometry but disagreed about arithmetic.
- Arithmetic truths are analytic by the first test.
- But how can logic be committed to objects?

Talk outline

Three principles

In the enquiry that follows, I have kept to three
fundamental principles:

> *always to separate the psychological from the*
> *logical, the subjective from the objective;*
> *never to ask for the meaning of a word in isolation,*
> *but only in the context of a proposition;*
> *never to lose sight of the distinction between*
> *concept and object.*

Anti-psychologism

- Psychologism: mathematical entities are mental
- Two arguments against this view:
    - Ideas are private, whereas numbers don't seem to be.
    - Some numbers have never been thought of.
- The *Grundlagen* is widely considered to have demolished psychologism.

Concept and object

- *Object*: entity referred to by a *singular term*
- *Concept*: entity referred to by a *predicate*
- Frege draws a sharp distinction between object and concept.
- The distinction between sense and reference was not clear at the time of writing *Grundlagen*.

Concepts

- From 'Frege is a logician', we can form first-level predicate 'x is a logician'
- First-level predicates pick out first-level concepts.
- From '$\forall x$ x is a logician', we can form second-level predicate '$\forall x X x$'
- Second-level predicates pick out second-level concepts.
- Concepts have *extensions*.
- *Objects* fall under first-level concepts.
- *First-level concepts* fall under second-level concepts.

Objects

- An object is anything that can be united with a subject to form a judgement.
- To be a singular term is to function linguistically in the correct manner.
- Frege's criterion for objecthood is *semantic*.
- This notion of objecthood is captured by the Context Principle.

The Context Principle

- The *context principle* instructs us 'never to ask for the meaning of a word in isolation but only in the context of a proposition'.
- Michael Dummett argues that the context principle marks a fundamental shift in philosophy:
  the *linguistic turn* (compare: Kant's Copernican turn).

> ... it is only in the context of a sentence that a word has meaning: the investigation therefore takes the form of asking how we can fix the senses of sentences containing words for numbers. (Dummett 1993: 5)

Talk outline

Implicit definition

- A major use of the Context Principle is to justify *implicit definition*.
- *Explicit* definitions define expressions in terms of previously understood expressions: 'vixen' means female fox.
- *Implicit* definition define expressions functionally: Jack the Ripper committed the Whitechapel murders.
- Explicit definitions *mention* the definiendum.
- Implicit definitions *use* it.
- Frege offered an implicit definition of number.

Counting

- Suppose I want to know whether the number of knives is the same as the number of forks.
- I could count the knives, then the forks, then compare.
- In Fregean terms, I would be assigning numbers of objects to the first-level concepts *knife on the table* and *fork on the table*.

One-one correspondence

- Frege (§63) notes that there is another way, suggested by Hume

  > When two numbers are so combined as that the one
  > has always an unit answering to every unit of the
  > other, we pronounce them as equal
  > (Treatise, bk 1, part iii, §1)

- We could pair the knives and forks off.
- This establishes a *one-one correspondence*.
- So the number of knives is the same as the number of forks.

Counting

- Frege thought that *all* counting was like this.
- When we count objects, we put them into one-one correspondence with an initial segment of the natural numbers.
- Say that there is a one-one correspondence between the knives and the numbers 1 to n.
- Say that there is a one-one correspondence between the forks and the numbers 1 to m.
- Then there is a one-one correspondence between the knives and forks if, and only if, n is m.

One-one correspondence

- One-one correspondence can be logically defined.
- Relation R is a one-one correspondence between the Fs and the Gs iff:
    1. No object bears R to more than one object, and
    2. no object is borne R by more than one object, and
    3. every F bears R to some G, and
    4. every G is borne R by some F.
- In symbols:

$$\forall x(Fx \rightarrow \exists! y(Gy \land Rxy)) \land$$
$$\forall y(Gy \rightarrow \exists! x(Fx \land Rxy))$$

Hume's Principle

- Let's say *there is* such a one-one correspondence:
$$\exists X(\forall x(Fx \rightarrow \exists!y(Gy \wedge Xxy)) \wedge$$
$$\forall y(Gy \rightarrow \exists!x(Fx \wedge Xxy)))$$

- And let's abbreviate this long expression as:
$$F \sim G$$

- We are now in a position to define numerical identity:
$$HP \quad NxFx = NxGx \leftrightarrow F \sim G$$

Abstraction Principles

- HP is an *abstraction principle*: it purports to give us semantic and epistemological access to abstract objects.
- Abstraction principles have the following shape:

$$\Sigma a = \Sigma b \leftrightarrow Eab$$

- Here, 'Σ' is a *term-forming* operator.
- It denotes a function from items in the range of the first-order variables to objects.
- 'E' expresses an equivalence relation over the items in the range of the first-order variables.

Abstraction Principles

- The equivalence relation E is already understood.
- The kinds of objects in its range are uncontroversial.
- The truth-conditions of instances of 'Eab' are unproblematic.
- The principle tells us that the truth-conditions of identity statements involving 'Σ' are coincident with these unproblematic ones.
- Overall, we can exploit the entities on the RHS to explain our knowledge of, and reference to, the entities on the LHS.

Hume's Principle

- In the case of HP, we can exploit our access to one-one correspondence in order to access numbers.
- The first-order variables in HP range over everyday objects.
- By coming to know the truth-conditions of instances of the RHS of HP, we can come to know those of the LHS.
- And if the LHS instances are true identity statements, then '*NxFx*' must be a genuine singular term.
- For Frege, to be an object is to be referred to by a singular term.

Talk outline

The significance of HP

- *Frege Arithmetic* is the theory built on second-order logic with HP as its only axiom.
- *Frege's Theorem* is the remarkable result that we can derive all of the second-order Peano axioms as theorems of Frege Arithmetic.
- Frege does not prove Frege's Theorem in *Grundlagen*.
- The proof was conjectured by Crispin Wright and proved by, amongst others, George Boolos (see his 'On the proof of Frege's Theorem', 1988).

Proving Frege's Theorem

- To prove Frege's Theorem, we need to show that all of the axioms of PA^2 are theorems of Frege Arithmetic.
- Let's sketch how this result might be proved.
- We also need to define the successor function. Frege instead defines a predecessor function:

$$Pmn =_{df}$$
$$\exists F \exists y (n = NxFx \wedge Fy \wedge m = Nx(Fx \wedge x \neq y))$$

- In words: m is the predecessor of n just if n is the number of Fs, for some F, and m is the number of Fs excluding one object.

Proof sketch

- Consider the following axiom of PA^2:

  Ax $\forall x \forall y(Sx = Sy \rightarrow x = y)$

- In terms of predecession, this can be expressed as:

  Ax$'$ $\forall x \forall y \forall z(Pxz \wedge Pyz) \rightarrow x = y))$

- To prove that Ax$'$ is a theorem of Frege Arithmetic, we need the following *Equinumerosity Lemma*:

  Eq $(F \sim G \wedge Fa \wedge Gb) \rightarrow F^{-a} \sim G^{-b}$

- Here, F^{-a} is *the Fs apart from a*.

Proof sketch

- Assume Pac and Pbc.
- By the definition of Pxy,

$$Pmn =_{df}$$
$$\exists F \exists y(n = NxFx \land Fy \land m = Nx(Fx \land x \neq y))$$

  we know that there will be concepts F, G and objects d and e such that:
  - $c = NxFx \land Fd \land a = NxF^{-d}x$
  - $c = NxGx \land Ge \land b = NxG^{-e}x$
- If $c = NxFx$ and $c = NxGx$, then $NxFx = NxGx$
- By HP, $F \sim G$
- By Eq, $F^{-d} \sim G^{-e}$. $[(F \sim G \land Fd \land Ge) \rightarrow F^{-d} \sim G^{-e}]$
- By HP, $NxF^{-d}x = NxG^{-e}x$.
- So $a = b$
- So $(Pac \land Pbc) \rightarrow a = b$
- So $\forall x \forall y \forall z(Pxz \land Pyz) \rightarrow x = y))$

The Julius Caesar Problem

- Frege rejected HP as a definition of number.
- Given the context principle, we have succeeded in settling the meaning of an expression just if we have settled the meaning of every sentence in which that expression features.
- But HP does not settle the meaning of all sentences involving number terms. It only settles those of the form '$NxFx = NxGx$'.

The Julius Caesar Problem

- Frege sums up the worry:

  we can never – to take a crude example – decide by means of our definitions whether any concept has the number Julius Caesar belonging to it, or whether that same familiar conqueror of Gaul is a number or not. (§56)

- HP allows us to settle the truth value of '$NxFx = NxGx$': it's true just if the Fs and Gs can be put in one-one correspondence).

- But it is silent as to the truth value of '$NxFx =$ Julius Caesar'.

- This problem is now known as the Julius Caesar problem.

Explicit definition

- In light of the Julius Caesar problem, Frege opts for an *explicit* definition of number:

    > *My definition is therefore as follows:*
    >
    > > *the Number which belongs to the concept F is the extension of the concept "equal to the concept F"*
    > > *(§68)*

- By this definition, numbers are *extensions of second-level concepts*.

- The extension of a second-level concept is all of the first-level concepts that fall under that second-level concept.

Explicit definition

- The extension of concept F is $\{x : Fx\}$'
$$Nx Fx =_{df} \{X : X \sim F\}$$

- 'The number of members of The Beatles' is a term picking out the set of all concepts equinumerous with the first-level concept *member of the Beatles*.

- One such first-level concept is *prime numbers less than or equal to 7*.

- There is a second-level concept under which only these and other 4-membered first-level concepts fall.

- This second-level concept has an extension, and its extension is identical to the number of members of The Beatles.

Extensions

- Does the explicit definition solve the Julius Caesar problem?
- 'I assume that it is known what the extension of a concept is' (§68, n.1).
- The explicit definition enjoys all of the technical benefits of HP, since
  $$NxFx =_{df} \{X : X \sim F\}$$
  straightforwardly entails:
  $$NxFx = NxGx \leftrightarrow F \sim G$$
- Frege came to spell out the details in *Grundgesetze der Arithmetik* (*Basic Laws of Arithmetic*, first volume published in 1893, second in 1903).
- Here, the problems with his definition in terms of extensions became apparent.