Philosophy of Mathematics
Neo-logicism

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25/10/18

Logicism

- We considered Frege's logicism in the *Grundlagen*.

  HP $NxFx = NxGx \leftrightarrow F \sim G$

- Frege Arithmetic: second-order logic + HP
- Frege's Theorem: Frege Arithmetic $\vdash PA^2$
- This does not settle the truth value of '$NxFx =$ Julius Caesar'.
- Instead, numbers are *extensions of second-level concepts*:

  '$NxFx$': $\{X : X \sim F\}$

- 'I assume that it is known what the extension of a concept is' (§68, n.1).

Talk outline

Basic Law V

Neo-logicism

Neo-Fregeanism

Conclusion

Basic Law V

- Numbers are the extensions of second-level concepts.
- How do extensions behave?
- *Extensional equivalence* is enshrined in *Grundgesetze* as *Basic Law V*:

  V $\hat{F} = \hat{G} \leftrightarrow \forall x(Fx \leftrightarrow Gx)$

- Basic Law V is contradictory.

Contradiction

V $\hat{R} = \hat{G} \leftrightarrow \forall x(Rx \leftrightarrow Gx)$

- Consider the first-level concept R whose extension is:

  1 $\{x : \exists G(x = \hat{G} \wedge \neg Gx)\}$

- Assume:

  2 $R(\hat{R})$

- So:

  3 $\exists G(\hat{R} = \hat{G} \wedge \neg G(\hat{R}))$

- By Basic Law V:

  4 $\neg R(\hat{R})$

- So:

  5 if $R(\hat{R})$ then $\neg R(\hat{R})$

- Exactly analogous reasoning gives:

  6 if $\neg R(\hat{R})$ then $R(\hat{R})$

- So:

  7 $R(\hat{R})$ if and only if $\neg R(\hat{R})$

Russell's Paradox

- Russell informed Frege in a letter of 16th June 1902:

  I find myself in complete agreement with you in all essentials. ... There is just one point where I have encountered a difficulty. You state that a function, too, can act as the indeterminate element (i.e. the variable). This I formerly believed, but now this view seems doubtful to me because of the following contradiction: Let w be a predicate which cannot be predicate of itself. Can w be predicated of itself? From each answer the opposite follows. (Selected Letters, Vol. I, p. 246)

Frege's reply

- Frege later wrote that, in light of Russell's discovery, his

  > *efforts to throw light on the questions surrounding the word 'number' and the words and signs for individual numbers seem to have ended in complete failure. (Posthumous Writings, p. 265)*

Neo-logicism

- Frege was right that his logicism was a failure.
- But many today are *neo-logicists*.
- Frege was essentially correct but wrong about Basic Law V.
- Instead, we should return to HP as our definition of number.
- This point was first made by Charles Parsons in his 1965 paper ‘Frege’s theory of number’ and worked out in detail in Crispin Wright’s 1983 *Frege’s Conception of Numbers as Objects*.
- The best place to start is his 2001 collection of papers co-authored with Bob Hale, *The Reason’s Proper Study*.

Talk outline

Hume's Principle

- Traditional Fregean logicism attempted to reduce mathematics to logic.
- For the neo-logicist, this task has 2 parts:
  (i) showing that HP is a logical truth; and
  (ii) showing that second-order logic is really logic.

Infinite domains

- HP is only satisfiable in infinite domains.

  HP $NxFx = NxGx \leftrightarrow F \sim G$
- Assume, for *reductio*, that HP is finitely satisfiable.
- Then there is some finite model $\mathcal{M}$ in which HP is true.
- Since $\mathcal{M}$ is finite, its domain $\mathcal{D}$ is of size k, for some finite k.
- HP maps equinumerous concepts to the same object and nonequinumerous concepts to distinct objects.
- So we need at least one object for each size of concept.

Infinite domains

- How many objects must there be in the domain?
  1. Concepts under which exactly 1 object fall are mapped to some object o_1 in $\mathcal{D}$.
  2. Concepts under which exactly 2 objects fall are mapped to some distinct object o_2 in $\mathcal{D}$, ...

  k Concepts under which exactly k objects fall are mapped to some distinct object o_k in $\mathcal{D}$.

  0 But there are also concepts under which 0 objects fall. These concepts will be mapped to some distinct object o_0.
- Hence, $\mathcal{D}$ contains $k+1$ objects.
- But $k = k+1$ only in infinite domains, so the size of $\mathcal{D}$ must be infinite. Contrary to our assumption.
- So HP is satisfiable only in infinite models.

Second-order logic

- Is second-order logic really logic?
- Consider the logical truths of second-order logic:
  1. $\exists X \forall x (Xx \leftrightarrow x \neq x)$
  2. $\exists X \forall x (Xx \leftrightarrow x = x)$
- The first requires the existence of the empty set.
- The second the existence of a universal set.
- But there's no commitment even to a two-membered set: $\exists X \exists x \exists y (Xx \wedge Xy \wedge x \neq y)$ is not a logical truth.

Analytic logicism

- Instead, neo-logicists attempt to reduce mathematics not to the *logical* but to the *analytic*.
- Frege seems to have equated logical and analytic truth.
- Neo-logicists attempt to pull the two apart: HP is *analytically*, though not *logically* true.
- This is the position put forward by (some time-slices of) Hale and Wright.

Is Hume's Principle analytic?

- In his 1997 paper 'Is Hume's Principle analytic?', Boolos gives 3 reasons to think that analytic logicism is false:

1. Analytic truths should not commit us to infinitely many objects.
2. Frege Arithmetic is equiconsistent with second-order arithmetic. But we don't know whether second-order PA is consistent.
3. By HP, there is an object – zero – corresponding to the size of the concept *being non self-identical*. Similarly, there is an object – antizero – corresponding to the size of the concept *being self-identical*.

Talk outline

Neo-Fregeanism

- For these reasons, many neo-logicists endorse an alternative view, *neo-Fregeanism*.
- E.g. Wright (1999) argues that HP can be stipulated as an implicit definition of the term-forming operator $Nx\ldots x\ldots$.
- Perhaps such a stipulation renders HP analytic, but that is not the primary concern.
- Neo-Fregeanism, understood this way, may have semantic and epistemological advantages.

Neo-Fregeanism

- There is no mystery about our ability to talk about or gain knowledge of HP.
- It is true by stipulation and implicitly defines our number talk.
- Analytic logicism leaves the semantic and epistemological questions open.
- If Frege Arithmetic is analytic, we still need a story about how we can access analytic truths.

A dilemma for neo-Fregeanism

- Robert Trueman (2015) raises an objection in his 'A dilemma for neo-Fregeanism'.
- What do the neo-Fregeans want from their stipulation? They want (at least) the following:
  (i) the truth of HP to be guaranteed; and
  (ii) the meaning of '$Nx\ldots x\ldots$' to be fixed.
- What does it mean to *stipulate* HP as an implicit definition?
  - a *subsentential* stipulation, of '$Nx\ldots x\ldots$'
  - a *sentential* stipulation, of '$NxFx = NxGx$'

Subsentential stipulation

- If we offer HP as a subsentential stipulation of '$Nx...x...$', we are stipulating the following:

  > '$Nx...x...$' is to refer to whatever function it needs to refer to for HP to be true.

- This sort of stipulation does fix the meaning of '$Nx...x...$'.
- It provides a criterion for deciding of a given function whether it is the function that '$Nx...x...$' picks out.

Subsentential stipulation

- Consider the 'Jack the Ripper' example:

  > 'Jack the Ripper' is to refer to the person who committed the Whitechapel murders.

- This stipulation allows us to decide of a given person whether they are the person referred to by 'Jack the Ripper'.
- They are if they committed the Whitechapel murders.

Guaranteed truth?

- But neo-Fregeans also want their stipulation to guarantee the truth of HP.
- Here, subsentential stipulation seems hopeless.
- There is no guarantee that ‘Jack the Ripper committed the Whitechapel murders’ is true.
- Perhaps no one committed them, or perhaps many people did.
- The stipulation guarantees the truth of:
  - If someone committed the Whitechapel murders, then Jack the Ripper did.

Carnap sentences

- In David Lewis's 1970 'How to define theoretical terms', he introduces the notion of a *Carnap sentence*.
- Consider theory T, which we can think of as the conjunction of its axioms.
- 'T' contains the theoretical term 't', so we'll write the sentence as '$T(t)$'.
- We can define 't' by stipulating that 't' refers to the thing that satisfies '$T(x)$'.
- This stipulation does not guarantee that '$T(t)$' is true.
- It guarantees the truth of T's Carnap sentence:

  $$\exists x Tx \rightarrow Tt$$

Carnap sentences

- Consider a theory with just the axiom 'Jack the Ripper committed the Whitechapel murders'.
- If we want that theory to implicitly define 'Jack the Ripper', we guarantee the truth of the Carnap sentence.
- Similarly, the *subsentential* stipulation of '$Nx\ldots x\ldots$' does not guarantee the truth of HP but only of the Carnap sentence:

$$\exists f \forall F \forall G (fxFx = fxGx \leftrightarrow F \sim G) \rightarrow \\ \forall F \forall G (NxFx = NxGx \leftrightarrow F \sim G)$$

Sentential stipulation

- Perhaps, then, the neo-Fregean should offer a *sentential* stipulation of HP:

  > '$NxFx = NxGx$' is to have the same truth-value as '$F \sim G$' on every assignment of values to 'F' and 'G'.

- This stipulation obviously guarantees the truth of HP.
- Does it fix a meaning for the term-forming operator '$Nx \ldots x \ldots$'?

Complexity

- Sentential stipulation tells us nothing at all about the complexity of '$NxFx = NxGx$'.
- All we are told is that instances of '$NxFx = NxGx$' have the same truth-value as corresponding instances of '$F \sim G$'.

An analogy

- Suppose I introduce a sentence '$F(Socrates)$' and fix its truth-value by stipulating that it is true.
- Are we allowed, on the basis of this stipulation, to take 'Socrates' as a singular term?
- No: we should then be allowed to substitute it for other names, but the stipulation is silent on the truth-value of '$F(Plato)$'.

An analogy

- Suppose I introduce to the sentence '$G(Socrates)$' by stipulating that 'G' is true of an object iff that object committed suicide.
- The truth-value of '$G(Socrates)$' is in part fixed by the referent of 'Socrates'.
- But the truth-value of '$F(Socrates)$' is not. In '$F(Socrates)$', 'Socrates' is not a relevant semantic unit.

Identity

- The usual response from neo-Fregeans is to insist that the sentence '$NxFx = NxGx$' *does* have structure.
- '$=$' is the identity sign.
- But the neo-Fregean is not allowed to make this response.
- The way in which we fix truth-value settles the role that the parts of that sentence play.
- If we do not treat '$NxFx$' as a semantic unit when fixing the meaning, then it plays no role in fixing the truth-value of the whole.
- And when we sententially stipulate HP, we are not treating '$NxFx$' as a semantic unit.

Dilemma

- Neo-Fregeans can offer either a sentential or a subsentential stipulation.
- The former guarantees the truth of HP but does not fix the referent of '$Nx...x...$'.
- The latter fixes the referent of '$Nx...x...$' but is silent on the truth of HP.
- Neo-Fregeans need both.
- So neo-Fregeans fail.

Talk outline

Conclusion

- Neo-logicists believe that Frege went wrong in abandoning HP as a definition of number.
- They come in two broad flavours: analytic logicists and neo-Fregan logicists.
- The former attempt to reduce arithmetic to the analytic but face problems with the ontological commitments of second-order logic and HP.
- The latter attempt to stipulate HP as an implicit definition of number.
- But they face a dilemma based on the nature of the stipulation.