Logicism

- We considered Frege’s logicism in the *Grundlagen*.
  \[ \text{HP} \quad NxFx = NxGx \leftrightarrow F \sim G \]
- Frege Arithmetic: second-order logic + HP
- Frege’s Theorem: Frege Arithmetic \( \vdash PA^2 \)
- This does not settle the truth value of ‘\( NxFx = \text{Julius Caesar} \)’.
- Instead, numbers are *extensions of second-level concepts*:
  \[ ‘NxFx’: \{X: X \sim F\} \]
- ‘I assume that it is known what the extension of a concept is’ (§68, n.1).
Talk outline

Basic Law V

Neo-logicism

Neo-Fregeanism

Conclusion
Basic Law V

- Numbers are the extensions of second-level concepts.
- How do extensions behave?
- *Extensional equivalence* is enshrined in *Grundgesetze* as *Basic Law V*:
  
  \[
  \forall \hat{F} = \hat{G} \iff \forall x (Fx \iff Gx)
  \]

- Basic Law V is contradictory.
Contradiction

\[ V \hat{R} = \hat{G} \iff \forall x (Rx \iff Gx) \]

- Consider the first-level concept \( R \) whose extension is:
  
  1 \( \{x : \exists G (x = \hat{G} \land \neg Gx)\} \)

- Assume:
  
  2 \( R(\hat{R}) \)

- So:
  
  3 \( \exists G (\hat{R} = \hat{G} \land \neg G(\hat{R})) \)

- By Basic Law \( V \):
  
  4 \( \neg R(\hat{R}) \)

- So:
  
  5 \( \text{if } R(\hat{R}) \text{ then } \neg R(\hat{R}) \)

- Exactly analogous reasoning gives:
  
  6 \( \text{if } \neg R(\hat{R}) \text{ then } R(\hat{R}) \)

- So:
  
  7 \( R(\hat{R}) \text{ if and only if } \neg R(\hat{R}) \)
Russell’s Paradox

- Russell informed Frege in a letter of 16th June 1902:

  I find myself in complete agreement with you in all essentials. ... There is just one point where I have encountered a difficulty. You state that a function, too, can act as the indeterminate element (i.e. the variable). This I formerly believed, but now this view seems doubtful to me because of the following contradiction: Let \( w \) be a predicate which cannot be predicate of itself. Can \( w \) be predicated of itself? From each answer the opposite follows. (Selected Letters, Vol. I, p. 246)
Frege’s reply

Frege later wrote that, in light of Russell’s discovery, his efforts to throw light on the questions surrounding the word ‘number’ and the words and signs for individual numbers seem to have ended in complete failure. (Posthumous Writings, p. 265)
Neo-logicism

- Frege was right that his logicism was a failure.
- But many today are *neo-logicists*.
- Frege was essentially correct but wrong about Basic Law V.
- Instead, we should return to HP as our definition of number.
- This point was first made by Charles Parsons in his 1965 paper ‘Frege’s theory of number’ and worked out in detail in Crispin Wright’s 1983 *Frege’s Conception of Numbers as Objects*.
- The best place to start is his 2001 collection of papers co-authored with Bob Hale, *The Reason’s Proper Study*. 
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Hume’s Principle

- Traditional Fregean logicism attempted to reduce mathematics to logic.
- For the neo-logicist, this task has 2 parts:
  (i) showing that HP is a logical truth; and
  (ii) showing that second-order logic is really logic.
Infinite domains

- HP is only satisfiable in infinite domains.
  \[\text{HP} \quad NxFx = NxGx \iff F \sim G\]
- Assume, for *reductio*, that HP is finitely satisfiable.
- Then there is some finite model \(\mathcal{M}\) in which HP is true.
- Since \(\mathcal{M}\) is finite, its domain \(D\) is of size \(k\), for some finite \(k\).
- HP maps equinumerous concepts to the same object and nonequinumerous concepts to distinct objects.
- So we need at least one object for each size of concept.
Infinite domains

- How many objects must there be in the domain?
  1. Concepts under which exactly 1 object fall are mapped to some object $o_1$ in $\mathcal{D}$.
  2. Concepts under which exactly 2 objects fall are mapped to some distinct object $o_2$ in $\mathcal{D}$, ...
  k. Concepts under which exactly $k$ objects fall are mapped to some distinct object $o_k$ in $\mathcal{D}$.
  0. But there are also concepts under which 0 objects fall. These concepts will be mapped to some distinct object $o_0$.

- Hence, $\mathcal{D}$ contains $k + 1$ objects.

- But $k = k + 1$ only in infinite domains, so the size of $\mathcal{D}$ must be infinite. Contrary to our assumption.

- So HP is satisfiable only in infinite models.
Is second-order logic really logic?

Consider the logical truths of second-order logic:

1. \( \exists X \forall x (Xx \leftrightarrow x \neq x) \)
2. \( \exists X \forall x (Xx \leftrightarrow x = x) \)

The first requires the existence of the empty set.

The second the existence of a universal set.

But there’s no commitment even to a two-membered set:
\( \exists X \exists x \exists y (Xx \land Xy \land x \neq y) \) is not a logical truth.
Instead, neo-logicists attempt to reduce mathematics not to the *logical* but to the *analytic*.

Frege seems to have equated logical and analytic truth.

Neo-logicists attempt to pull the two apart: HP is *analytically*, though not *logically* true.

This is the position put forward by (some time-slices of) Hale and Wright.
Is Hume’s Principle analytic?

In his 1997 paper ‘Is Hume’s Principle analytic?’, Boolos gives 3 reasons to think that analytic logicism is false:

1. Analytic truths should not commit us to infinitely many objects.

2. Frege Arithmetic is equiconsistent with second-order arithmetic. But we don’t know whether second-order PA is consistent.

3. By HP, there is an object – zero – corresponding to the size of the concept being non self-identical. Similarly, there is an object – antizero – corresponding to the size of the concept being self-identical.
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For these reasons, many neo-logicists endorse an alternative view, *neo-Fregeanism*.

E.g. Wright (1999) argues that HP can be stipulated as an implicit definition of the term-forming operator $N\ldots x\ldots$.

Perhaps such a stipulation renders HP analytic, but that is not the primary concern.

Neo-Fregeanism, understood this way, may have semantic and epistemological advantages.
There is no mystery about our ability to talk about or gain knowledge of HP.

It is true by stipulation and implicitly defines our number talk.

Analytic logicism leaves the semantic and epistemological questions open.

If Frege Arithmetic is analytic, we still need a story about how we can access analytic truths.

What do the neo-Fregeans want from their stipulation? They want (at least) the following:

(i) the truth of HP to be guaranteed; and
(ii) the meaning of ‘$Nx\ldots\times\ldots$’ to be fixed.

What does it mean to stipulate HP as an implicit definition?

- a *subsentential* stipulation, of ‘$Nx\ldots\times\ldots$’
- a *sentential* stipulation, of ‘$Nx F x = N x G x$’
If we offer HP as a subsentential stipulation of ‘\( \text{\textit{N}x...x...} \)’, we are stipulating the following:

‘\( \text{\textit{N}x...x...} \)’ is to refer to whatever function it needs to refer to for HP to be true.

This sort of stipulation does fix the meaning of ‘\( \text{\textit{N}x...x...} \)’.

It provides a criterion for deciding if a given function whether it is the function that ‘\( \text{\textit{N}x...x...} \)’ picks out.
Consider the ‘Jack the Ripper’ example:

‘Jack the Ripper’ is to refer to the person who committed the Whitechapel murders.

This stipulation allows us to decide of a given person whether they are the person referred to by ‘Jack the Ripper’.

They are if they committed the Whitechapel murders.
Guaranteed truth?

- But neo-Fregeans also want their stipulation to guarantee the truth of HP.
- Here, subsentential stipulation seems hopeless.
- There is no guarantee that ‘Jack the Ripper committed the Whitechapel murders’ is true.
- Perhaps no one committed them, or perhaps many people did.
- The stipulation guarantees the truth of:
  - If someone committed the Whitechapel murders, then Jack the Ripper did.
Carnap sentences

- In David Lewis’s 1970 ‘How to define theoretical terms’, he introduces the notion of a Carnap sentence.
- Consider theory $T$, which we can think of as the conjunction of its axioms.
- ‘$T$’ contains the theoretical term ‘$t$’, so we’ll write the sentence as ‘$T(t)$’.
- We can define ‘$t$’ by stipulating that ‘$t$’ refers to the thing that satisfies ‘$T(x)$’.
- This stipulation does not guarantee that ‘$T(t)$’ is true.
- It guarantees the truth of $T$’s Carnap sentence:
  \[ \exists x T x \rightarrow T t \]
Consider a theory with just the axiom ‘Jack the Ripper committed the Whitechapel murders’.

If we want that theory to implicitly define ‘Jack the Ripper’, we guarantee the truth of the Carnap sentence.

Similarly, the *subsentential* stipulation of ‘$Nx \ldots x \ldots$’ does not guarantee the truth of HP but only of the Carnap sentence:

\[
\exists f \forall F \forall G (fxFx = fxGx \leftrightarrow F \sim G) \rightarrow \\
\forall F \forall G (NxFx = NxGx \leftrightarrow F \sim G)
\]
Perhaps, then, the neo-Fregean should offer a *sentential* stipulation of HP:

\[ NxFx = NxGx \] is to have the same truth-value as \( F \sim G \) on every assignment of values to \( F \) and \( G \).

This stipulation obviously guarantees the truth of HP.

Does it fix a meaning for the term-forming operator \( Nx...x... \)?
Sentential stipulation tells us nothing at all about the complexity of \( NxFx = NxGx \).

All we are told is that instances of \( NxFx = NxGx \) have the same truth-value as corresponding instances of \( F \sim G \).
Suppose I introduce a sentence ‘$F(Socrates)$’ and fix its truth-value by stipulating that it is true.

Are we allowed, on the basis of this stipulation, to take ‘Socrates’ as a singular term?

No: we should then be allowed to substitute it for other names, but the stipulation is silent on the truth-value of ‘$F(Plato)$’.
An analogy

▶ Suppose I introduce to the sentence ‘\(G(Socrates)\)’ by stipulating that ‘\(G\)’ is true of an object iff that object committed suicide.
▶ The truth-value of ‘\(G(Socrates)\)’ is in part fixed by the referent of ‘Socrates’.
▶ But the truth-value of ‘\(F(Socrates)\)’ is not. In ‘\(F(Socrates)\)’, ‘Socrates’ is not a relevant semantic unit.
The usual response from neo-Fregeans is to insist that the sentence ‘$NxFx = NxGx$’ does have structure.

‘$=$’ is the identity sign.

But the neo-Fregean is not allowed to make this response.

The way in which we fix truth-value settles the role that the parts of that sentence play.

If we do not treat ‘$NxFx$’ as a semantic unit when fixing the meaning, then it plays no role in fixing the truth-value of the whole.

And when we sententially stipulate HP, we are not treating ‘$NxFx$’ as a semantic unit.
Dilemma

- Neo-Fregeans can offer either a sentential or a subsentential stipulation.
- The former guarantees the truth of HP but does not fix the referent of ‘\( N_\ldots \cdot \ldots \)’.
- The latter fixes the referent of ‘\( N_\ldots \cdot \ldots \)’ but is silent on the truth of HP.
- Neo-Fregeans need both.
- So neo-Fregeans fail.
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Conclusion
Neo-logicists believe that Frege went wrong in abandoning HP as a definition of number.

They come in two broad flavours: analytic logicists and neo-Fregan logicists.

The former attempt to reduce arithmetic to the analytic but face problems with the ontological commitments of second-order logic and HP.

The latter attempt to stipulate HP as an implicit definition of number.

But they face a dilemma based on the nature of the stipulation.