Philosophy of Mathematics
Structuralism

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Neo-Fregeanism

- Last week, we considered recent attempts to revive logicism.
- *Analytic* logicists try to show that HP is an analytic truth.
- But HP is only satisfiable in infinite domains.
- *Neo-Fregean* logicism tries to stipulate HP as an implicit definition of number.
- But stipulation must be sentential or subsentential.
- The former fails to fix a reference for the term-forming operator.
- The latter fails to guarantee that HP is true.

Talk outline

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Implicit definition

- We've been discussing *implicit definition*.
- A *functionalist* about the mental may define 'pain' in the following way:
  - Pain is caused by stubbing your toe and pain makes you wince and pain is unpleasant and ...
  - x is caused by stubbing your toe and x makes you wince and x is unpleasant and ...
- 'Pain' is anything that satisfies this open sentence.
- It is multiply realizable.

Geometry

- We may take a similar approach in the philosophy of mathematics:
  - Given any two points, exactly one line can be drawn which passes through them.
  - Any line can be indefinitely extended.
  - Given any two points, exactly one line can be drawn which passes through them AND Any line can be indefinitely extended AND ...
  - Given any two points, exactly one x can be drawn which passes through them AND Any x can be indefinitely extended AND ...
- This open sentence may implicitly define ‘line’.
- Many entities could satisfy the open sentence.

Hilbert

- In a letter to Frege, Hilbert writes:

  Every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and the basic elements can be thought of in any way one likes. If in speaking of my points, I think of some system of things, e.g. the system love, law, chimney-sweep ... and then assume all my axioms as relations between these things, then my propositions, e.g., Pythagoras' theorem, are also valid for these things ... any theory can always be applied to infinitely many systems of basic elements. (see Frege, Philosophical and mathematical correspondence)

Structuralism

- Further evidence for this *structuralist* line of thought: relative consistency proofs.
- To show that theory Θ_1 is consistent *relative to* Θ_2:
    1. Interpret the nonlogical primitives of the former in the language of the latter;
    2. show that the results are theorems of the latter.
- Hilbert initiated this method of proof, which is now common.
- Virtually all of mathematics can be modelled in set theory.
- This all suggests a form of structuralism.

What numbers could not be

- Natural numbers can be identified with set-theoretic objects.
- Frege did something like this.
- Paul Benacerraf, in ‘What Numbers Could Not Be’ (1965), notes that this can be done in infinitely many ways.
- Without loss of generality, let’s think about 2 ways.

Two identifications

I Zermelo:

$$0 = \varnothing$$
$$1 = \{\varnothing\}$$
$$2 = \{\{\varnothing\}\}$$
$$3 = \{\{\{\varnothing\}\}\}$$
$$\vdots$$

II von Neumann:

$$0 = \varnothing$$
$$1 = \{\varnothing\}$$
$$2 = \{\varnothing, \{\varnothing\}\}$$
$$3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$$
$$\vdots$$

Identification problem

- Which of I and II is to be preferred?
- We can define successor, addition and multiplication on both.
- They make precisely the same sentences true.
- But they cannot *both* be true.
- By I, $2 = \{\{\varnothing\}\}$; by II, $2 = \{\varnothing, \{\varnothing\}\}$
- By transitivity of identity, $\{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}\}$. False
- The correct conclusion seems to be that neither is correct.
- Rather, numbers are identified *structurally*: '2' picks out a certain position, which can be instantiated by $\{\{\varnothing\}\}$ or $\{\varnothing, \{\varnothing\}\}$.

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Systems

- Following Hilbert, a *system* is a collection of objects with relations between them:

  > *any theory can always be applied to infinitely many systems of basic elements*

- A *natural number* system is a countably infinite collection of objects with a designated initial object and a successor relation satisfying the Peano axioms.
- E.g. Arabic numerals or an infinite sequence of temporal points.
- A *Euclidean* system is three collections of objects, one called 'points', one 'lines' and one called 'planes', with relations between them satisfying the Euclidean axioms.

Structures

- A *structure* is the abstract form of a system.
- It abstracts away from all properties of the system other than the structural relations that hold between the objects.
- The Arabic numerals and the infinite sequence of temporal points share the *natural number structure*.

Varieties of structuralism

- All structuralists contend that mathematics is the science of structure.
- They differ as to the nature of these structures.
- The main distinction is between *platonist* and *nominalist* structuralists.
- *Platonist* structuralists, most notably Stewart Shapiro, Charles Parsons and Michael Resnik, hold that the structures exist.
- *Nominalist* structuralists, most notably Geoffrey Hellman and Charles Chihara, hold that the structures can be eliminated.
- We'll consider the nominalist option next week.

Platonist structuralism

- Universals can be *platonic* or *aristotelian*.
- *Platonic* universals are not spatio-temporally located: they are abstract entities.
- *Aristotelian* universals are spatio-temporally located: they are located in their instances.
- The defender of aristotelian universals has a metaphysical advantage.
- They need not accept anything abstract over and above the concrete instances.
- But they say that one universal can be wholly present at different places at the same time.
- And they hold that two universals can occupy the same place at the same time.

Ante rem and *in re* structuralists

- Platonist structuralists generally come in two sorts:
  - *Ante rem* structuralism: structures are *platonic* universals
  - *In re* structuralism: structures are *aristotelian* universals
- *In re* structuralists have to say that the structures are ontologically dependent on their systems.
- Destroy the systems and you destroy the structures.
- But while this may have some non-mathematical appeal, it seems problematic here.
- We'll proceed with *ante rem* structuralism as our paradigm example of Platonist structuralism.

Advantages of Platonist structuralism

- The Platonist structuralist can read mathematical sentences at face value.
    1. There are at least 2 primes less than 7.
    2. There are at least 2 cities smaller than Paris.
- They can accommodate Benacerraf's insights.
- They have an explanation for relative consistency proofs.
- Mathematical objects are *abstracta* of a familiar sort: universals.
- Shapiro has a sophisticated epistemological story.

Shapiro's stratified epistemology

Abstraction We are able to discern small, instantiated patterns. E.g. we recognise the *2-pattern* in all systems consisting of just 2 objects. By attending to such tokens, we apprehend the types.

Projection We can gain knowledge of larger structures, e.g. the *10,000-pattern*, by recognising the order of smaller abstractions, and projecting further.

Description Projection will only get us as far as denumerable infinities, however. To deal with larger structures, we can use our powers of description. If we know the coherence of such descriptions, we gain knowledge of the structures described.

Problems for projection

- It seems plausible that we are capable of something like abstraction.
- Fraser MacBride (1997) casts doubt on whether projection and description can deliver knowledge of structures.
- It seems plausible that we do in fact project our abstractions about smaller patterns to larger cases.
- E.g. we discern the *2-*, *3-* and *4-patterns* and believe that this progression will continue indefinitely.
- But such belief does not suffice for *knowledge* without some sort of *warrant* or *justification*.

Problems for projection

- We need more than the *descriptive* claim that we do project.
- We need the *normative* claim that we are *right* to project.
- Perhaps we derive *general* knowledge about all structures from *particular* knowledge about certain structures.
- But what premises would allow such a derivation?
- It seems they would have to be general statements themselves.
- Further, we face rule-following worries: how do we know that we should not be following some less standard rule?

Problems for description

- Here, Shapiro echoes Hilbert: if we establish the consistency of an axiomatization, we can establish the existence of the structure described.
- Shapiro doesn't use Hilbert's notion of consistency but the wider notion of *coherence*.
- 'Coherence' is a primitive notion that gets explicated in terms of set-theoretic satisfiability:

  if a description is coherent, it is guaranteed that the description is satisfied by at least one structure

Problems for description

- If we know that an axiomatization is *coherent*, it is satisfied by at least one structure.
- If we also know that it is *categorical*, then it is satisfied by at most one structure.
- So we can know there is *exactly one* structure corresponding to the description.
- But how do we come to know coherence and categoricity?
- These notions get spelt out in set-theoretic terms.
- The question of their justification gets pushed back to mathematical theory.
- Again, these are the very things we're out to justify.
- The issue is how finite creatures access the infinite. But this access is as mysterious in the case of set theory as it is anywhere.

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Identity

- Recall Quine's slogan 'no entity without identity'.
- Shapiro agrees:

  > *Quine's thesis is that within a given theory, language, or framework, there should be definite criteria for identity among its objects. There is no reason for structuralism to be the single exception to this. (1997: 92)*

- If positions in structures are genuine objects, then Shapiro owes us an account of their *identity conditions*.

Identity

- Shapiro writes that

  > *The essence of a natural number is its relation to other natural numbers ... there is no more to the individual numbers 'in themselves' than the relations they bear to each other. (pp 72-3)*

- This suggests a version of the *Identity of Indiscernibles* for mathematical objects:

  > If x and y share just the same intrastructural relations to other items, then $x = y$

- But now he faces a problem: there are many examples of distinct mathematical objects that share the same intrastructural properties.

Automorphisms

- An *isomorphism* between two sets is a structure-preserving one-to-one map between those sets.
- An *automorphism* on a set is an isomorphism from that set to itself.
- Identity is the *trivial automorphism*.
- Some mathematical structures have *nontrivial automorphisms*.

The additive integers

- The *additive inverse* of an integer a is the number that, added to a, yields 0.
- Intuitively, it is the *opposite* of the number.
- E.g. the additive inverse of 2 is -2; that of -4 is 4; 0 is its own additive inverse.
- Consider the nontrivial automorphism f that maps each integer to its additive inverse.
- This preserves all additive structure: for any objects x, y, z such that $x + y = z$, $f(x) + f(y) = f(z)$.
- E.g. $4 + 2 = 6$

  $-4 + (-2) = -6$.

The complex plane

- Recall that a *complex* number is one that can be expressed as $a + bi$, where a and b are real and $i^2 = -1$.
- Consider the nontrivial automorphism f that maps each complex number of the form $a + bi$ to $a - bi$.
- E.g. $4 + 5i$ is mapped to $4 - 5i$

  $8 - 10i$ is mapped to $8 - (-10)i = 8 + 10i$.
- This preserves *all* structure: for any objects x, y, z *and any operation* * such that $x * y = z$, $f(x) * f(y) = f(z)$.
- E.g. $(8 + 4i) \times (2 + 3i) = 4 + 32i$

  $(8 - 4i) \times (2 - 3i) = 4 - 32i$.
- And there are many examples in the Euclidean plane.

Haecceities

- There are entities mathematicians usually take to be distinct which are structurally identical.
- Perhaps the structuralist can appeal to *haecceities*.
- The haecceity of an object A is *the property of being identical to A*.
- The number 4 uniquely instantiates the property of *being the number 4*. The number -4 does not.
- This gives up the idea that there is no more to mathematical objects than their structural properties.

Reductionism

- Perhaps structuralists should offer a reductionist thesis not of objects but *properties*.
- There is no more to the *properties* of numbers than their structural relations.
- But mathematical objects cannot be so reduced.
- Structurally indiscernible but distinct *objects* are no threat to this view.

Properties

- Is property reductionism tenable?
- It seems that numbers have non-structural properties:
  - being abstract
  - being my favourite number
  - being the number of planets
- Property structuralism entails that *objects* cannot be so reduced.
- So objects are a separate category of existent.
- So we're back to traditional Platonism.
- Thus Fraser MacBride: either bad news ($i = -i$) or old news (traditional Platonism).

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- Structuralists claim that mathematics is the science of structure.
- Benacerraf's insights and relative consistency proofs suggest that some form of structuralism is true.
- The most plausible *Platonic* structuralism is *ante rem* structuralism, where mathematical entities are platonic universals.
- Shapiro's is an influential version of this view.
- But it faces epistemological problems and identity problems, given the existence of *nontrivial automorphisms*.