Philosophy of Mathematics
Nominalism

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8/11/18

Last week

- *Ante rem* structuralism accepts mathematical structures as Platonic universals.
- We posed problems for Shapiro's stratified epistemology in terms of abstraction, projection and description.
- We also posed an *identity* problem: some structures admit of *nontrivial automorphisms*.
- E.g. The integers 4 and -4 share all *additive* intrastructural properties.
- Worse, i and $-i$ share *all* intrastructural properties.

Talk outline

Modal structuralism

Indispensability

Conclusion

Nominalist structuralism

- Given these problems, some have endorsed *nominalist* structuralism.
- We'll focus on Geoffrey Hellman's *modal* structuralism, as presented in *Mathematics without Numbers* (1989).
- The central idea is that we can enjoy the benefits of Platonist structuralism but without the ontological commitment.
- We don't need *actual* structures, we only need *possible* ones.
- We talk about structures only within the scope of modal operators.

Arithmetic

- An ω-sequence is a set of objects that
  (i) has the cardinality of the natural numbers; and
  (ii) is ordered as the natural numbers are usually ordered.
- Consider some arithmetic sentence S.
- Hellman would paraphrase S:

  $$\Box\forall X(X \text{ is an } \omega\text{-sequence} \rightarrow S \text{ holds in } X)$$
- In words: necessarily, if x is an ω-sequence, then S is true in x.
- This is the *hypothetical* component.
- Note this is second-order.

Vacuity

- There is an immediate vacuity problem.
- The nominalist structuralist does not want to be committed to mathematical objects.
- These are the very things that caused issues for Shapiro.
- But, for all we know, the universe may not contain enough physical objects for all of mathematics.
- E.g. if the universe contains only $10^{100,000}$ objects, then there won't be any ω-sequences.

Vacuity

- Even if we are convinced that the universe contains enough objects for arithmetic, there may be some limit.
- There will be branches of mathematics that require more.
- If there are not enough objects, the antecedent of the hypothetical component will be false.
- So the conditional will be vacuously true.
- For this reason, Hellman introduces the *categorical* component:

  $$\Diamond \exists X (X \text{ is an } \omega\text{-sequence})$$

- In words: it is possible that an ω-sequence exists.

The categorical component

- The categorical component guarantees that, if the hypothetical component is true, it is non-vacuously true.
- There's one wrinkle. Both components are expressed in second-order S5. But it had better be S5 without the *Barcan Formula*:

BF	$\Diamond\exists X FX \rightarrow \exists X \Diamond FX$
BF′	$\Diamond\exists X(X \text{ is an } \omega\text{-sequence}) \rightarrow \exists X\Diamond(X \text{ is an } \omega\text{-sequence})$
CC	$\Diamond\exists X(X \text{ is an } \omega\text{-sequence})$
$\therefore$	$\exists X\Diamond(X \text{ is an } \omega\text{-sequence})$

- Hellman obviously doesn't want this.

More precisely

- PA^2 is the conjunction of the axioms of second-order PA.
- The hypothetical modal paraphrase of S is:

  H $\Box\forall X\forall f(PA^2 \to S)^X(s/f)$
- And the categorical paraphrase:

  C $\Diamond\exists X\exists f(PA^2)^X(s/f)$
- Here, s/f is the result of replacing s with f.
- ϕ^X is the result of relativizing the quantifiers in ϕ to X.
- '0' can be defined.
- Hellman also offers paraphrases of sentences of real analysis and set theory.

Modality

- Overall, mathematical sentences are elliptical for longer sentences in second-order S5 (without BF).
- They have a hypothetical component to achieve this, and a categorical component to avoid vacuity.
- The account avoids Shapiro's metaphysical burden.

Modality

- But the nature of the invoked modality must be explained.
- Is it metaphysical, logical, mathematical?
- Hellman says logical, but now there's a problem.
- Logical modality usually gets explicated in set-theoretic terms, but that is to let abstract objects back in.
- Instead, he refuses to explicate the logical modality at all, and leaves it primitive.

Ontology vs ideology

- Quine says that a theory carries an *ontology* and an *ideology*
- The ontology consists of the entities which the theory says exist.
- The ideology consists of the ideas expressed within the theory using predicates, operators, etc.
- Ontology is measured by the number of entities postulated by a theory.
- Ideology is measured by the number of primitives.
- It is often thought that ideological economy has epistemological benefits.
- A theory with fewer primitives is likely to be more unified, which may aid understanding.
- Hellman has increased ideology in order to reduce ontology.

Problems for modal structuralism

- Hellman also faces epistemological and semantic worries.
- Epistemologically, it is not at all obvious that facts about primitive logical modality are any more accessible than mathematical facts.
- Further, possibility presumably implies consistency. But we have no idea about the consistency of the cases being considered.
- Semantically, Hellman loses the nice uniform semantics offered by e.g. Shapiro.

Applications

- Further, modal structuralism has little to say about the *applications* of mathematics.
- Hellman supposedly gives a framework in which mathematics can be understood without ontological commitment.
- But this is not how mathematicians understand their utterances.
- Hellman could advance his view as *prescriptive* but that seems implausible.

Talk outline

The indispensability argument

- The indispensability argument was put forward by Quine and Putnam as an argument for platonism about mathematical objects.
- Put simply, it has the following form:
  1. We ought to be ontologically committed to all and only those entities that are indispensable to our best scientific theories.
  2. Mathematical entities are indispensable to our best scientific theories.

  ∴ 3 We ought to be ontologically committed to mathematical entities.

The indispensability argument

- Why believe the first premise?
  1. We ought to be ontologically committed to all and only those entities that are indispensable to our best scientific theories.
- Quine and Putnam argue for it using a combination of *naturalism* and *holism*:

  Naturalism Philosophy is continuous with science. It is neither prior to nor privileged over science.

  Holism Theories are confirmed or disconfirmed as wholes.
- Let's look at these in turn.

Naturalism

- Our concern is *methodological* naturalism:

  the only authoritative standards in the philosophy of mathematics are those of the natural sciences

- It has precursors in the empiricist tradition, e.g. the logical positivists, Mill and Hume.
- This is accepted in one form or another by most philosophers.
- One prominent kind of opponent is the *intuitionist*.
- It is far from clear that all of science can be reconstructed in intuitionistic terms.

Naturalism

- Naturalism tells us nothing directly about the *sorts* of entities we should accept.
- As it stands, we shouldn't accept *ghosts* but this is only contingently so.
- If best theory included ghosts, we would have reason to believe in them.
- Naturalism implies that we should believe in *only* the entities that feature in our best scientific theories.
- Whether we should also believe in *all* of them is unclear.

Holism

- But the indispensability argument only goes through if we accept *all* such entities.
- Otherwise, mathematics may not be part of the naturalistic commitments.
- Holism rules out this possibility.
- Our concern is with *confirmational* holism as opposed to Quine's *semantic* holism.
- Theories are confirmed as wholes, so confirmation of the mathematical part is guaranteed.

Maddy on indispensability

- Nominalists need to undermine the indispensability argument.
- Next week, we'll see Harty Field's fictionalist attack on the second premise.
- For now, let's consider Penelope Maddy's influential attacks on the first premise.
- She seeks to undermine the first premise by undermining the combination of its motivations: naturalism and holism.
- See her paper 'Indispensability and Practice' (1992).

Holism and naturalism

- According to naturalism, we should take scientific practice seriously.
- But scientists have different attitudes to the parts of well-confirmed theories.

  > *Logically speaking, this holistic doctrine is unassailable, but the actual practice of science presents a very different picture. Historically, we find a wide range of attitudes toward the components of well-confirmed theories, from belief to grudging tolerance to outright rejection. (Maddy 1992: 280)*

- Her main illustration is atomic theory.

Atomic theory

- Atomic theory was well-confirmed as early as 1860 in light of fruitfulness, systematic advantages, etc.
- Still, many scientists were sceptical until the turn of the century.
- Ingenious experiments were then performed to directly verify the existence of atoms.
- And only the directly verifiable consequences of atomic theory were believed.
- Confirmation was taken to spread only so far.
- Hence, naturalism and holism can pull apart.

Truth and utility

- The lesson is that scientists do not take treat their theories homogeneously.
- Some parts are taken to be true; others merely useful.
- E.g. the analysis of water waves assumes that the water is infinitely deep.
- Fluid dynamics treats matter as continuous, ignoring that it is made of atoms.
- These assumptions are *essential*: our best theories would not work without them.
- Again, naturalism and holism lead us in different directions.

Independence

- We know that there are interesting set-theoretic claims that are *independent* of ZFC, e.g. Continuum Hypothesis (CH).
- Reactions to these independent claims vary.
- Some are willing to assert bivalence; others claim that they are indeterminate.
- There are many suggestions of additional set-theoretic axioms that would either imply CH or imply $\neg$CH.
- And these discussions occur independently of scientific practice.
- Indispensability implies that set theorists consider the scientific upshots of their axioms.
- This again goes against practice.

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- Issues such as the identity problem may lead us to doubt *ante rem* structuralism.
- If we want to maintain structuralism, we may move to a *modal* structuralism.
- But this view relies on a primitive modality, which seems as inaccessible as much of mathematics.
- And as a nominalist view of mathematics, it faces application and indispensability worries.
- We have seen, though, that there are reasons to doubt indispensability.
- Next week, we explore these further as part of Hartry Field's fictionalism.