Philosophy of Mathematics
Fictionalism

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Last Week

- *Nominalist* structuralists try to enjoy the benefits of Platonist structuralism but without the ontological commitment.
- Hellman's *modal* structuralism is an example.
- But Hellman faces problems with his primitive notion of logical modality.
- He also needs a response to the indispensability argument.
- We saw Maddy's attack on this argument.

Talk outline

Field on indispensability

Problems for Field

Conclusion

Fictionalism

- Harty Field, in his *Science Without Numbers* (1980) argues that mathematical objects do not exist.
- He takes mathematical sentences at face value.
- Singular terms that purport to refer to numbers are empty.
- So mathematical sentences are false.
- Field's first task is to undermine the indispensability argument.

Dispensability

- Recall the indispensability argument:
  1. We ought to be ontologically committed to all and only those entities that are indispensable to our best scientific theories.
  2. Mathematical entities are indispensable to our best scientific theories.

  ∴ 3 We ought to be ontologically committed to mathematical entities.
- Field attacks the second premise.
- Mathematics *is* dispensable to science.
- Mathematized theories are *conservative* over their nominalised counterparts.

Conservativeness

- A theory Θ_2 is a *conservative extension* of a theory Θ_1 iff
  1. the language of Θ_2 extends the language of Θ_1;
  2. every theorem of Θ_1 is a theorem of Θ_2; and
  3. any theorem of Θ_2 that is not a theorem of Θ_1 is not expressible in the language of Θ_1.
- Intuitively, they agree where they overlap.
- Anything new couldn't even be *said* before.

Nominalism and conservativeness

- Let M be a mathematized theory, and N be its nominalistic counterpart.
- Where M contains mathematical terms, N contains nominalistically respectable counterparts.
- M is a conservative extension of N iff
  1. the language of M extends that of N;
  2. every theorem of N is a theorem of M; and
  3. any theorem of M that is not a theorem of N is not expressible in the language of N.

Nominalism and conservativeness

- Field claims that mathematized theories conservatively extend their nominalistic counterparts.
- We could restrict ourselves to the nominalistic theories, and we wouldn't be missing anything.
- That's why maths is *dispensable*. But why is it still *useful*?
- Mathematical inference can be *faster*.

Nominalism and conservativeness

- For example, let N_C be some nominalistic sentence that follows from nominalistic sentences $N_1, ..., N_n$.
- We could derive N_C from $N_1, ..., N_n$ directly, but the derivation could be long and time-consuming.
- In many cases, it will be easier to ascend to the mathematical counterparts $M_1, ..., M_n$, derive the mathematical M_C and finally descend to the nominalistic N_C.

Field's project

- Field's project has 2 parts:
  (i) produce the nominalistic counterparts to mathematized theories;
  (ii) prove that the mathematical theories are conservative over their nominalistic counterparts.
- (i) is a huge task: different nominalistic theories will be required each time.
- Field intends to instigate a research programme, but begins with Newtonian gravitational theory.
- The techniques can be used more widely.

Newtonian gravitational theory: ontology

- The basic ontology of Newtonian gravitational theory consists of ordered quadruples of real numbers.
- These are used to specify spatial coordinates in four-dimensional space.
- Field instead uses regions formed by space-time points themselves.

Newtonian gravitational theory: vocabulary

- The vocabulary includes functors expressing functions whose values are numbers, e.g. 'the gravitational potential of x'.
- Field uses comparative predicates, e.g. 'the difference in gravitational potential between x and y is less than that between z and w'.
- When completed, the semantic value is a truth-value rather than a number.
- These comparative claims are the *concrete counterparts* of abstract claims.

Newtonian gravitational theory

- Field needs *bridge laws* to explain the utility of Newtonian theory.
- He also needs *representation theorems* to justify them.
- A simple example connects numerical claims about distance with comparative claims about points.
- E.g. his nominalised theory contains the predicates

Bet(xyz)	x is a point on a line segment whose end points are y and z
Cong($xyzw$)	the line segment with end points x and y is congruent to the line segment with end points z and w

Representation theorems

- Field proves that there is a *distance* function d that maps pairs of space-time points to real numbers:
  1. for any points x, y and z, Bet(yxz) iff $d(x, y) + d(y, z) = d(x, z)$
  2. for any points x, y, z and w, Cong($xyzw$) iff $d(x, y) = d(z, w)$
- If d represents distance, then this theorem shows that claims about congruence and between-ness are equivalent to claims about distance.
- These allow us to pass from comparative claims about space-time points to abstract numerical claims about distances, and *vice versa*.

Representation theorems

- Field (pp. 61–91) provides other representation theorems.
- E.g. there is a 'spatio-temporal co-ordinate function' mapping space-time points into quadruples of reals.
- And there are 'gravitational potential' and 'mass density' functions mapping space-time points to intervals of reals.
- So abstract claims about gravitational potential and mass density are equivalent to comparative claims about space-time points.

Talk outline

Field on indispensability

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Initial worries

- Some initial concerns:
  1. Field presupposes *substantivalism* about space-time: the view that space-time exists as a concrete entity. A controversial view
  2. Field has chosen a scientific theory that is peculiarly amenable to nominalisation. It is generally thought that e.g. quantum mechanics would be much more challenging.
  3. He hasn't even come close to establishing that mathematics is dispensable to *all* scientific theories.

Conservativeness

($\models$) M is a conservative extension of N iff

1. the language of M extends that of N;
2. every sentence that is a **logical consequence** of N is a **logical consequence** of M; and
3. any sentence that is a **logical consequence** of M that is not a **logical consequence** of N is not expressible in the language of N.

($\vdash$) M is a conservative extension of N iff

- $1'$ the language of M extends that of N;
- $2'$ every sentence **provable** in N is **provable** in M; and
- $3'$ any sentence **provable** in M that is not **provable** in N is not expressible in the language of N.

Semantic conservativeness

- The semantic notion invokes *logical consequence*, which is defined in terms of models in the usual post-Tarskian way:

  $\Gamma \models \phi$ iff every model of Γ is a model of ϕ
- Models are ordered pairs of domain and valuation function.
- But domains are sets, which are mathematical objects.
- In this way, abstract entities have reappeared in the metalogic.
- This problem also beset Hellman.

Syntactic conservativeness

- Syntactic conservativeness isn't as *obviously* mathematical.
- But it invokes *provability*:

  ϕ is provable from Γ iff there is proof of ϕ from Γ using only the rules of a standard Natural Deduction system
- This definition invokes 'proofs', but what are these?
- Proof *tokens* or *types*?
- The former are concretely inscribed proofs.
- But there aren't enough of them: many proofs have never been written down.
- The latter are abstract types.
- Here we have all the proofs we need, but they are abstract once again.

Consistency

- At some points, Field suggests defining the semantic notion in terms of *consistency*:

  > ϕ is a logical consequence of Γ iff Γ and $\neg\phi$ are logically inconsistent

- We now have a notion of logical consistency to explain, and the usual explanation is modal:

  > some sentences are logically consistent iff it is possible that they can all be true together

- We now have a modal notion to deal with.
- As with Hellman, what sort of modality is it?
- We could take the modality as primitive.
- Like Hellman, that increases ideology and raises access problems.

Conservativeness and incompleteness

- Stewart Shapiro, in his ‘Conservativeness and incompleteness’ (1983), poses problems for Field based on Gödel’s incompleteness results.
- Since the natural number series can be modelled within Field’s nominalistic gravitational theory (NT), the theory is incomplete.
- In particular, $NT \nvdash Con(NT)$.
- But, if we ascend to the mathematized counterpart of NT and use set theory, we can prove consistency.
- So we have that $NT + M \vdash Con(NT)$.
- So mathematics is not conservative!

Second-order theories

- But Field claimed to have proved conservativeness, so what's going on?
- Field proved *semantic* conservativeness.
- If we are working in a second-order theory, the situation is coherent: $\Gamma \models \phi \not\to \Gamma \vdash \phi$.
- But first-order logic is complete so syntactic non-conservativeness implies semantic non-conservativeness.
- Second-order logic isn't obviously available to the nominalist.
- It quantifies over sets or properties.

Representation theorems

- The incompleteness argument crucially relied on the existence of *representation theorems* to take us from NT to $NT + M$.
- If we deny that there are such theorems, we can enjoy the benefits of first-order logic *and* have conservativeness.
- But this is a high price to pay: Field needed representation theorems for his arguments against indispensability.
- So ultimately he faces a dilemma: either adopt second-order logic or give up representation theorems.

Talk outline

Conclusion

- We have seen two arguments against the indispensability argument.
- Maddy attacks its first premise from considerations about mathematical practice.
- Field attacks its second premise by arguing that mathematical objects are dispensable because conservative.
- But can he characterise conservativeness in a way that suits his purposes?
- And incompleteness also poses serious problems for him.