Philosophy of Mathematics – Lecture 1

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Mathematics and Naturalism

Flat-footed Realism: Numbers, sets/classes, etc. exist. They are abstract, outside space and time, causally inert. Yet we can talk, and know, about them.

Flat-footed realism is prone to standard naturalist worries.

Paul Benacerraf (1973) "Mathematical Truth" *Journal of Philosophy*

But naturalists can't just dismiss mathematics. For (a) mathematics is highly successful and (b) science requires mathematics.

Cf the Indispensability Argument; lecture 6 or 7.

Moreover, the prospects for *reduction* to the physical look bleak: for one, there just aren't enough physical things.

Other Puzzles

There are also lots of weird things about mathematics that flat-footed realism leaves unexplained:

- Mathematics is useful in science. Why would that be so if it's about some third realm, disconnected from the physical world?

  Hartry Field (1980) *Science Without Numbers*. Chapters 1-3

- Mathematical objects exist necessarily. But how could that be? How could something's nature guarantee its existence?

  Stephen Yablo (2002) "Abstract Objects: A Case Study" *Philosophical Issues*

- Mathematical objects have (almost?) only extrinsic properties.
- Mathematicians care about reduction – e.g. of arithmetic to set theory. But they don't care which reduction is correct.

  Paul Benacerraf (1965) "What Numbers Could Not Be" *Philosophical Review*

- Mathematicians study conflicting principles about what seem like the same things. But they don't care which is true.

Tentative Plan

Logicism (weeks 2-4): Mathematics is reducible to logic.

We won't cover *Intuitionism* (or *Constructivism*), which is revisionary; or *Formalism*, which is on the syllabus for the Mathematical Logic paper.

Structuralism (weeks 5-6): Mathematics is the study of structures, which physical systems do or could instantiate.

Fictionalism (weeks 7-8): Mathematics is a useful fiction. Its usefulness is *cognitive* and requires something *truth-like*, which mathematical practice tracks.

Axiomatizing Arithmetic

Arithmetic is the theory of the natural numbers (i.e. $0, 1, 2, \ldots$). We can characterize these using the constant '0' and the function 'successor':

1. 0 is a number ('0' is well-defined)
2. S is a function from numbers to numbers ('S' is well-defined)

1.-4. say that numbers can be constructed by starting with 0 and repeatedly adding 1.

3. Different numbers have different successors
   $\forall x \forall y (S(x) = S(y) \rightarrow x = y))$
4. 0 is not the successor of any number
   $\forall x S(x) \neq 0$
5. If 0 has a property, and that property is preserved by the successor function, then every number has that property
   $\Big(F(0) \& \forall x \big(F(x) \rightarrow F(S(x))\big)\Big) \rightarrow \forall x Fx$

5. adds that *all* the numbers can be constructed that way.

1.-5. are the Peano Axioms/Peano Arithmetic (PA). Can define normal arithmetical expressions in PA: 1 is the successor of 0, + is the function satisfying $\forall x(x + 0 = x)$ and $\forall x \forall y (x + S(y)) = S(x + y)$.

Even though the axiomatization was discovered by Dedekind, not Peano.

First vs Second-Order Logic

First-order PA (PA^1) treats Induction (axiom 5.) as a scheme: any way of replacing 'F' with an open formula yields an axiom.

Second-order PA (PA^2) treats Induction as a single axiom, prefixed with a universal quantifier $\forall F$, quantifying into predicate position.

First-order logic, the familiar kind, only allows quantification into object position.

The difference: there might be 'features' for which the language has no predicates (simple or complex).

In fact there must be. There are only countably many predicates in the language. But there is at least one feature for every set of natural numbers. So some features are not expressed.

Controversial whether second-order logic 'counts' as logic.

Two Theorems

Categoricity (Dedekind): Any two models of PA^2 are isomorphic – there is a structure-preserving pairing between them.

Formally: Let $<D_a, 0_a, S_a>$ and $<D_b, 0_b, S_b>$ be models. Then there is a function $f : D_a \rightarrow D_b$ that is 1-1 and onto, with $f(0_a) = f(0_b)$ and $\forall x f(s_a(x)) = s_b(f(x))$.

So PA^2 is (semantically) complete: if S is in the language of arithmetic, S is either true in every model of PA^2 or false in every such model.

Incompleteness (Gödel): For any (recursively) axiomatized theory containing PA^1, there is a sentence S in the language of arithmetic, such that the theory proves neither S nor $\neg S$.

So second-order logic cannot be axiomatized.

Some Models of PA

$0, 1, 2, \ldots$

$25, 250, 2500, \ldots$

My thought that p, my thought that I'm thinking that p, my thought that I'm thinking that I'm thinking that p, ...

$\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, $\{\{\{\emptyset\}\}\}$, ...

$\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, ...

That is: $0, \{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots$

Reading for Next Week

- Gottlob Frege (1884) *Foundations of Arithmetic* §§ 46-47, 55-72