Philosophy of Mathematics – Lecture 2

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Frege's First Insight: Numbers belong to Concepts (§§46-47)

Things, or states of affairs, can't be assigned numbers in a determinate way: 4 companies = 500 men; 3 courses = 1 meal; 5 items of clothing = 1 outfit; etc.

But can determinately assign numbers to concepts: how many Fs?

Maybe we can count objects – but the results will ususally be counterintuitive: when you have 2 shoes, you also have 1 pair of shoes → at least 3 objects.

Frege's First Proposal (§§55-61)

Frege notes that you can logically define 'numerical quantifiers':

$\exists_0 xFx =_{def} \neg\exists xFx$

$\exists_1 xFx =_{def} \exists x(Fx \wedge \forall y(Fy \rightarrow y = x))$

$\exists_2 xFx =_{def} \exists x\exists y(Fx \wedge Fy \wedge x \neq y \wedge \forall z(Fz \rightarrow (z = x \vee z = y)))$

...

$\exists_{n+1} xFx =_{def} \exists x(Fx \wedge \exists_n y(Fy \wedge y \neq x))$

This may be familiar from Russell's theory of definite descriptions

Can then try to formulate arithmetical claims – e.g. '$5 + 3 = 8$' might become the claim $\forall F\forall G\big(\exists_5 xFx \wedge \exists_3(Gx \wedge \neg Fx) \rightarrow \exists_8(Fx \vee Gx)\big)$. Easy to see that this would be a logical truth.

Frege doesn't do this. Problem: $5 + 3 = 9$ becomes $\forall F\forall G\big(\exists_5 xFx \wedge \exists_3(Gx \wedge \neg Fx) \rightarrow \exists_9(Fx \vee Gx)\big)$. But this will be *true* if, say, there are only 4 objects!

The Problem:

> It is an illusion that we have defined 0 and 1; in reality, we have only fixed the sense of the phrases 'the number 0 belongs to' and 'the number 1 belongs to'; but we have no authority to pick out the 0 and 1 here as self-subsistent objects. (§56)

Compare: a definition of 'to do A for X's sake' won't tell us what a *sake* is.

We have defined second-order concepts; not (yet) objects.

Why is this a problem? Frege's arguments

- In natural language, number terms sometimes take object-position: 'the number of moons of Jupiter is 4'.
- We cannot settle whether Julius Caesar is a number.
- We cannot prove that if n belongs to F, and so does m, then $n = m$.

Not clear Frege is right about this. See Thomas Hofweber (2005) "Number Determiners, Numbers, and Arithmetic" *Philosphical Review*; Friederike Moltmann (2013) "Reference to Numbers in Natural Language" *Philosophical Studies*

Other problem:

- If there are only n things, we have $\forall F(\exists_{n+1} xFx \leftrightarrow \exists_{n+2} xFx)$. But then it's not clear that the two concepts are distinct.

In the *Principia*, Russell and Whitehead postulate the axiom of infinity to solve this. The need for this axiom is one of the main reasons why their system is thought to fail at reducing mathematics to logic.

Frege's Second Proposal (§§62-67)

What's needed to define a singular term? Can't just define some sentences in which it occurs. Instead, need to explain all the identity statements featuring that term.

Hume's Principle (HP): $\#(F) = \#(G)$ iff $F \approx G$.

'The number of Fs is identical to the number of Gs iff the Fs and the Gs are *equinumerous* (i.e. can be paired up).'

As Frege notes (§§70-72), we can define $F \approx G$ in second-order logic as $\exists H\Big(\forall x\big(Fx \rightarrow \exists_1 yH(x,y)\big) \wedge \forall y\big(Gy \rightarrow \exists_1 xH(x,y)\big)\Big)$

Direction Principle (DIR): $dir(a) = dir(b)$ iff $a \parallel b$

'The direction of l is identical to the direction of m iff l and m are parallel.'

is a function from first-order concepts to objects. HP ensures that the same object is associated with two concepts only if they fall under the same numerical quantifier – so there is a 1-1 correlation between these objects and the numerical quantifiers defined above.

This should feel a bit fishy – how can we get something like this out of what looks like a definition? A diagnosis: we can define # however we want – but it's a further claim that there really is a function meeting that definition. HP both gives the definition *and* asserts that it's well-defined; once you see that, it no longer looks so trivial. See George Boolos (1997) 'Is Hume's Principle Analytic?', reprinted in his (1998) *Logic, Logic, Logic*.

What have we gained?

- Numbers are now at the object level.
- This guarantees that there are infinitely many objects. Start with an empty concept, e.g. 'being non-self identical'. By HP, this gets a number, call it 0. Then we have a new concept, 'being identical to 0'. This isn't empty, so, by HP, gets a different number, call it 1. Then we have a new concept, 'being identical to 0 or 1'. This isn't equinumerous with the previous two, so gets a new number, call it 2. And so on.
- In fact, we get Frege's Theorem holds: in second-order logic, the Peano Axioms follow from HP.

Hypothesized by Parsons (1965) "Frege's Theory of Numbers"; shown in detail by Crispin Wright (1983) *Frege's Conception of Numbers as Objects*. Heck (1995) "The Development of Arithmetic in Frege's *Grundgesetze der Arithmetik*" shows that Frege himself proved essentially this.

But Frege isn't satisfied with DIR and HP:

> [This definition] does not, for instance, decide for us whether England is the same as the direction of the Earth's axis. . . [It] says nothing as to whether the proposition "the direction of a is identical to q" should be affirmed or denied, except for the one case where q is given in the form "the direction of b". (§66)

Compare the complaint about the first proposal that it doesn't settle whether Julius Caesar is a number. Hence the label 'the Julius Caesar problem'.

HP places a constraint on the function #. But there can be many functions satisfying that constraint and (a) some of them will map concepts to 'normal' objects, and (b) different ones will map the same concepts to different objects.

Frege's Third Proposal (§§68-69)

Frege's solution: identify numbers with *extensions*.

$\#(F) = \epsilon(\text{'equinumerous with F'}) = \{G : F \approx G\}$

So the numbers are the extensions of the numerical quantifiers.

Easy to see that this gives us HP. But questions arise. Small ones:

- Why *these* extensions?
- Do numbers have all the properties had by these extensions?

And the big one: why think the existence of extensions is any less problematic than that of numbers?

In *Grundgesetze*, Frege proposes to understand extensions via:

Basic Law V (BLV): $\epsilon(F) = \epsilon(G)$ iff $\forall x(Fx \leftrightarrow Gx)$

'The extension of F is identical to the extension of G iff the same things are Fs as are Gs.'

Puzzling: why doesn't the Julius Caesar problem arise for BLV?

Frege was aware of this – and may well have thought the only advantage of BLV was dialectical. See e.g. Richard Heck (2005) "Julius Caesar and Basic Law V" *dialectica*.

Anyways, BLV is inconsistent. It leads to naive set theory, which postulates a set for every condition we can name. But the condition 'not being a member of oneself' can't generate a set.

A diagnosis: BLV requires there to be as many extensions, hence as many objects, as there are concepts. But, by Cantor's theorem, there must be more concepts than objects.

For next week, read Bob Hale and Crispin Wright (2001) "Introduction" to their *The Reason's Proper Study*