Philosophy of Mathematics
Frege’s logicism

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Frege and Kant

- Kant took arithmetic and geometry to be bodies of synthetic *a priori* truths.
- Frege agreed about geometry, but disagreed about arithmetic.
- Frege took the test for the synthetic to be *dependence on intuition*.
- By this test, he thought arithmetic was analytic.
- To show this, he had to show how it was possible to be introduced to numbers independently of intuition.
Counting

Let’s suppose I want to know whether the number of knives on a table is the same as the number of forks.

I could count the knives, and then the forks, and compare the results.

In Fregean terms, I would be assigning numbers to the first-level concepts *knife on the table* and *fork on the table*. 
But, Frege (§63) notes that there is another way, suggested by Hume: ‘When two numbers are so combined as that the one has always an unit answering to every unit of the other, we pronounce them as equal’ (Treatise, bk 1, part iii, §1).

We could determine whether the number of knives was the number of forks by pairing them off.

If we do this, and there are none left over, then there is the same number of each.

We have established a *one-one correspondence* between the knives and the forks.

If there is a one-one correspondence between the knives and the forks, then the number of knives is the same as the number of forks; otherwise not.
Indeed, Frege thought that all counting was like this. When we counted the knives and forks, we were putting them into one-one correspondence with an initial segment of the natural numbers. Say that there is a one-one correspondence between the knives and the numbers 1 to $n$. And say that there is a one-one correspondence between the forks and the numbers 1 to $m$. Then there is a one-one correspondence between the knives and forks if, and only if, $n$ is $m$. 
One-one correspondence

- One-one correspondence can be logically defined.
- A relation $R$ is a one-one correspondence between the $F$s and the $G$s iff the following hold:
  1. The relation is one-one, that is, no object bears $R$ to more than one object, and no object is borne $R$ by more than one object.
  2. Every $F$ bears $R$ to some $G$ and every $G$ is borne $R$ by some $F$.
- In symbols:
  
  $$\forall x (Fx \rightarrow \exists! y (Gy \land Rxy)) \land$$
  $$\forall y (Gy \rightarrow \exists! x (Fx \land Rxy))$$
Hume’s Principle

- Now let’s say that, between the $F$s and $G$s, *there is* such a one-one correspondence:
  \[
  \exists R(\forall x(Fx \rightarrow \exists! y(Gy \land Rxy)) \land \\
  \forall y(Gy \rightarrow \exists! x(Fx \land Rxy))
  \]

- And let’s abbreviate this long expression as:
  \[F \sim G\]

- We are now in a position to define numerical identity:
  \[
  \text{HP} \quad NxFx = NxGx \iff F \sim G
  \]
Abstraction Principles

- HP is an *abstraction principle*: it purports to give us semantic and epistemological access to abstract objects.
- Abstraction principles have the following shape:
  \[ \Sigma a = \Sigma b \leftrightarrow Eab \]
- Here, ‘\(\Sigma\)’ is a *term-forming* operator: it denotes a function from items in the range of the first-order variables to objects.
- ‘\(E\)’ expresses an equivalence relation over the items in the range of the first-order variables.
Abstraction Principles

- It is assumed that the equivalence relation $E$ is already understood and that the kinds of objects in its range are uncontroversial.
- The truth-conditions of instances of ‘$Eab$’ are thought to be unproblematic.
- The principle tells us that the truth-conditions of identity statements involving ‘$\Sigma$’ are coincident with these unproblematic ones.
- Overall, we can exploit our knowledge of, and reference to, the entities described on the RHS in order to explain our knowledge of, and reference to, the entities described on the LHS.
Hume’s Principle

- In the case of HP, we can exploit our access to one-one correspondence in order to access numbers.
- The first-order variables in HP range over everyday objects: there is no general problem of referring to these or of gaining knowledge of their one-one correspondence.
- By coming to know the truth-conditions of instances of the RHS of HP, we can come to know those of the LHS.
- And if the LHS instances are true identity statements, then ‘$NxFx$’ must be a genuine singular term.
- And remember that, for Frege, there is no more to being an object than to be the sort of thing referred to be a singular term.
An *explicit* definition is one that defines an expression in terms of previously understood expressions, e.g. ‘vixen’ means female fox.

An *implicit* definition is a functional definition that defines an expression in terms of its role, e.g. Jack the Ripper is whoever committed the Whitechapel murders.

It may be that nobody committed the Whitechapel murders, or that many people did. But, if someone committed the Whitechapel murders, then that person is being defined as ‘Jack the Ripper’.

Crucially, in explicit definition, we *mention* the definiendum; in implicit, we *use* it.

HP is an *implicit* definition of number.
The Context Principle

- Recall that the context principle warns us to never to ask for the meaning of a word in isolation, but only in the context of a proposition.
- One use of the context principle that we saw last week was to argue against psychologism.
- Another is to legitimise implicit definition: if we can only ask for the meaning of a word in a proposition, then it is unsurprising that definitions are often implicit.
Remarkably, if we consider a theory which is built on second-order logic and has HP as its only axiom, we can derive all of the second-order Peano axioms as theorems.

Call this theory *Frege Arithmetic* and the result *Frege’s Theorem*.

Frege does not prove Frege’s Theorem in *Grundlagen*.

The proof was conjectured by Crispin Wright and proved by, amongst others, George Boolos (see his ‘On the proof of Frege’s Theorem’, 1988).
Proving Frege’s Theorem

To prove Frege’s Theorem, we need to show that all of the axioms of $\mathcal{PA}^2$ are theorems of Frege Arithmetic.

Let’s sketch how this result might be proved.

Let’s define ‘0’ in the following way:

$$0 =_{df} \mathcal{N}x \ x \neq x$$

We also need to define the successor function. Frege instead defines a predecessor function: a number $m$ is one less than a number $n$ if a concept that has the number $m$ is one object short of being a concept with the number $n$.

$$Pmn =_{df} \exists F \exists y (n = \mathcal{N}xFx \land Fy \land m = \mathcal{N}x(Fx \land x \neq y))$$

In words: $m$ is the predecessor of $n$ just if $n$ is the number of $F$s, for some $F$, and $m$ is the number of $F$s excluding one object.
Proof sketch

- One axiom of $\text{PA}^2$ states that 0 is not the successor of any number. We'll take that to mean that 0 has no predecessor.
- We could express this as:
  \[ \text{Ax } \neg\exists zP(z, Nx \neq x) \]
- We proceed by *reductio*. Suppose, for *reductio*, that:
  \[ 1 \exists zP(z, Nx \neq x) \]
- By the definition of predecession we have that:
  \[ 2 \exists F \exists z \exists y (Nx \neq x = NxFx \land Fy \land z = Nx(Fx \land x \neq y)) \]
- Therefore:
  \[ 3 \exists F \exists y (Nx \neq x = NxFx \land Fy) \]
- But, as Frege argues (*Grundlagen*, §75), if the number of $F$s is 0, then there is nothing that is $F$. So there is no such $y$. Contradiction.
- The other axioms can be proved in a similar way.
In spite of its power, Frege rejected HP as a definition of number.

Given the context principle, we have succeeded in settling the meaning of an expression just if we have settled the meaning of every sentence in which that expression features.

But HP does not settle the meaning of all sentences involving number terms. It only settles those of the form ‘\(NxFx = NxGx\)’. 

The Julius Caesar Problem
The Julius Caesar Problem

- Frege sums up the worry:
  
  *we can never – to take a crude example – decide by means of our definitions whether any concept has the number Julius Caesar belonging to it, or whether that same familiar conqueror of Gaul is a number or not.* (§56)

- HP allows us to settle the truth value of ‘\( NxFx = NxGx \)’: it’s true just if the \( F \)s and \( G \)s can be put in one-one correspondence).

- But it is silent as to the truth value of ‘\( NxFx = \text{Julius Caesar} \)’.

- This problem is now known as the Julius Caesar problem.
In light of the Julius Caesar problem, Frege opts for an explicit definition of number:

My definition is therefore as follows:

the Number which belongs to the concept F is the extension of the concept “equal to the concept F”

(§68)

By this definition, numbers are extensions of second-level concepts. Recall that the extension of a second-level concept is all of the first-level concepts that fall under that second-level concept.
We can think of extensions as sets and write ‘\( \{ x : Fx \} \)’ for the extension of the concept \( F \), i.e. the set of things falling under the concept \( F \). Frege is offering the following definition:

\[
NxFx =_{df} \{ X : X \sim F \}
\]

E.g. consider the term ‘the number of members of The Beatles’. By this definition, the term picks out the set of all concepts equinumerous with the first-level concept *member of the Beatles*.

One such first-level concept is *prime numbers less than or equal to 7*.

There is a second-level concept under which only these and other 4-membered first-level concepts fall.

This second-level concept has an extension, and its extension is identical to the number of members of The Beatles.
Extensions

- Does the explicit definition solve the Julius Caesar problem? Frege’s answer is somewhat unsatisfactory: ‘I assume that it is known what the extension of a concept is’ (§68, n.1).
- The explicit definition enjoys all of the technical benefits of HP, since
  \[
  NxFx =_{df} \{X : X \sim F\}
  \]
  straightforwardly entails:
  \[
  NxFx = NxGx \iff F \sim G
  \]
- Frege came to spell out the details in *Grundgesetze der Arithmetik* (*Basic Laws of Arithmetic*, first volume published in 1893, second in 1903).
- Here, the problems with his definition in terms of extensions became apparent.
Basic Law V

- Frege thought that extensions were governed by the *extensional equivalence*:
  \[ \{ x : Fx \} = \{ x : Gx \} \iff \forall x (Fx \leftrightarrow Gx) \]

- This principle gets enshrined in *Grundgesetze* as Basic Law V:
  \[ \forall F \forall G (\{ x : Fx \} = \{ x : Gx \} \iff \forall x (Fx \leftrightarrow Gx)) \]

- An immediate problem arises: Basic Law V is another implicit definition, so how about Julius Caesar?

- Frege’s answer is that, unlike *Grundlagen*, we are now working in a formal language that does not contain a name for Julius Caesar.

- This merely delays the problem, however: if we want arithmetic to be applicable, we’ll want to be able to count Julius Caesar and at this point he needs a name.
But there is a much more serious problem with Basic Law V: we can derive Russell’s Paradox from it. Here’s how.

Consider the concept $R$ whose extension is:

1. $\{x : \exists G (x = \{x : Gx\} \land \neg Gx)\}$

Now assume:

2. $R(\{x : Rx\})$

Then:

3. $\exists G (\{x : Rx\} = \{x : Gx\} \land \neg G(\{x : R\})$)

Then, by Basic Law V:

4. $\forall F \forall G (\{x : Fx\} = \{x : Gx\} \leftrightarrow \forall x (Fx \leftrightarrow Gx))$

5. if $R(\{x : Rx\})$ then $\neg R(\{x : Rx\})$

Exactly analogous reasoning gives:

6. if $\neg R(\{x : Rx\})$ then $R(\{x : Rx\})$

So:

7. $R(\{x : Rx\})$ if and only if $\neg R(\{x : Rx\})$
Russell’s Paradox

Russell informed Frege of the paradox in a letter of 16th June 1902:

I find myself in complete agreement with you in all essentials. ... There is just one point where I have encountered a difficulty. You state that a function, too, can act as the indeterminate element (i.e. the variable). This I formerly believed, but now this view seems doubtful to me because of the following contradiction: Let w be a predicate which cannot be predicate of itself. Can w be predicated of itself? From each answer the opposite follows. (Selected Letters, Vol. I, p. 246)
Frege’s reply

Frege later wrote that, in light of Russell’s discovery, his efforts to throw light on the questions surrounding the word ‘number’ and the words and signs for individual numbers seem to have ended in complete failure. (Posthumous Writings, p. 265)
Neo-Fregean Logicism

- Frege was right that his logicism was a failure.
- But many philosophers of maths today endorse *neo-logicism*. This is the view that Frege was essentially correct but wrong to endorse Basic Law V. Instead, we should return to HP as our definition of number, and answer the Julius Caesar problem.
- This point was first made by Charles Parsons in his 1965 paper ‘Frege’s theory of number’ but worked out in detail in Crispin Wright’s 1983 *Frege’s Conception of Numbers as Objects*.
- The best place to start is his 2001 collection of papers co-authored with Bob Hale, *The Reason’s Proper Study*.
- Next week, we turn to neo-logicism.