Philosophy of Mathematics
Neo-logicism

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Neo-logicism

- Neo-logicists hold that Frege was broadly correct but went wrong when he rejected HP as an implicit definition of number.
- Recall that *Frege Arithmetic* is a theory built on second-order logic with HP as its only axiom.
- Recall *Frege's Theorem* that the axioms of second-order Peano arithmetic are theorems of Frege Arithmetic.

Hume's Principle

- Traditional Fregean logicism attempted to reduce mathematics to logic.
- For the neo-logicist, this task has 2 parts: (i) showing that HP is a logical truth; (ii) showing that second-order logic is really logic.
- We'll take the 2 parts in turn.
- First, the central problem with taking HP to be a logical truth is that it is only satisfiable in infinite domains.

Proof

HP $NxFx = NxGx \leftrightarrow F \sim G$

- Assume, for *reductio*, that HP is satisfiable in a finite domain.
- Then there is some finite model $\mathcal{M}$ in which HP is true.
- Since $\mathcal{M}$ is finite, its domain $\mathcal{D}$ is of size k, for some finite k.
- Recall HP maps equinumerous concepts to the same object and nonequinumerous concepts to distinct objects.
- How many objects must there be in the domain?

Proof

- Concepts under which exactly 1 object fall are mapped to some object in $\mathcal{D}$,
- concepts under which exactly 2 objects fall are mapped to some distinct object in $\mathcal{D}$, ...
- concepts under which exactly k objects fall are mapped to some distinct object in $\mathcal{D}$.
- But there are also concepts under which 0 objects fall.
- So there is some other object in $\mathcal{D}$ to which these concepts are mapped.
- Hence, $\mathcal{D}$ contains $k + 1$ objects.
- But $k = k + 1$ only in infinite domains, so the size of $\mathcal{D}$ must be infinite. Contrary to our assumption.
- So HP is satisfiable only in infinite models.

Logical truths

- It is generally thought that logic is *ontologically innocent*, i.e. it should not commit us to the existence of objects.
- If the truth of HP requires the existence of objects, it is hard to call it a logical truth.
- Classically, $\exists x\ x = x$ is a logical truth, on the assumption that the domains of models must be non-empty.
- But even this is often thought to be too much, hence the popularity of inclusive logics, in which domains may be empty.
- HP is of course much worse than this: it requires the existence of infinitely many objects.

Second-order logic

- The same problem affects the second part of the neo-logicist's thesis: that second-order logic is logic.
- The difference comes out, however, when we look to the logical truths of second-order logic:
  1. $\exists X \forall x (Xx \leftrightarrow x \neq x)$
  2. $\exists X \forall x (Xx \leftrightarrow x = x)$
- The first requires the existence of the empty set; the second the existence of a universal set (though this will differ between models).
- But is this so much commitment? There's no commitment even to a two-membered set: $\exists X \exists x \exists y (Xx \wedge Xy \wedge x \neq y)$ is not a logical truth.

Continuum Hypothesis

CH $2^{\aleph_0} = \aleph_1$ (in words: the cardinality of the power set of the first infinite cardinal is the first uncountable cardinal).

- Let $N(X)$ be a second-order formula satisfied just if X is equinumerous with the natural numbers, and $R(Y)$ a second-order formula satisfied just if Y is equinumerous with the real numbers.
- Let $X < Y$ and $X \leq Y$ abbreviate the usual second-order renderings of 'X has a smaller size than Y' and 'X has a size smaller than or equal to Y', respectively.
- See Stewart Shapiro's *Foundations Without Foundationalism* (§5.1.2) for the details of these definitions.

S $\forall X \forall Y \forall Z((N(X) \wedge R(Y) \wedge X < Z) \rightarrow Y \leq Z)$

- S contains purely second-order logical vocabulary.
- S is logically true iff CH is true.

Second-order logic

- Some people, e.g. John Etchemendy (see *The Concept of Logical Consequence*) believe that this is an example of *overgeneration*.
- A definition of logical truth overgenerates when it finds some sentence logically true which intuitively isn't.
- There will be some other sentence S', which is logically true iff CH is false.
- Either S or S' is logically true, Etchemendy thinks. But neither should be, since intuitively the first 'expresses' CH and the latter $\neg$CH.

Analytic logicism

- Instead, neo-logicists attempt to reduce mathematics not to the *logical* but to the *analytic*.
- Frege seems to have equated logical and analytic truth, but neo-logicists attempt to pull the two apart: HP is *analytically*, though not *logically* true.
- This is the position put forward by (some time-slices of) Hale and Wright.

Is Hume's Principle analytic?

- In his 1997 paper 'Is Hume's Principle analytic?', Boolos gives 3 reasons to think that analytic logicism is false:

1. Analytic truths should not commit us to infinitely many objects any more than logical truths should.
2. We know that Frege Arithmetic is equiconsistent with second-order arithmetic. But we don't know whether second-order PA is consistent, so we don't know whether Frege Arithmetic is. If we do not *know* that HP is consistent, we are not in a position to call it analytic.
3. By HP, there is an object – zero – corresponding to the size of the concept *being non self-identical*. Similarly, there is an object – antizero – corresponding to the size of the concept *being self-identical*. But, in standard set theories, there is no universal set. Analytic truths should not conflict with our best mathematical theories.

Neo-Fregeanism

- For these reasons, many neo-logicists endorse an alternative view, *neo-Fregeanism*.
- E.g. Wright (1999) argues that HP can be stipulated as an implicit definition of the term-forming operator $Nx \ldots x \ldots$.
- Perhaps such a stipulation renders HP analytic, but that is not the primary concern.
- Neo-Fregeanism, understood this way, may have semantic and epistemological advantages.

Neo-Fregeanism

- There is no mystery about our ability to talk about or gain knowledge of HP.
- It is true by stipulation and implicitly defines our number talk, so we can use it to convert our talk and knowledge of equinumerosity into talk and knowledge of numbers.
- Analytic logicism, by contrast, leaves the semantic and epistemological questions open.
- If Frege Arithmetic is analytic, we still need a story about how we can refer to and gain knowledge of numbers.

A dilemma for neo-Fregeanism

- Robert Trueman (2015) raises an objection in his 'A dilemma for neo-Fregeanism'.
- What do the neo-Fregeans want from their stipulation? They want (at least) the following: (i) the truth of HP to be guaranteed; (ii) the meaning of '$Nx...x...$' to be fixed.
- What does it mean to *stipulate* HP as an implicit definition? It could mean one of two things.
- We could be offering a *subsentential* stipulation, of '$Nx...x...$'.
- We could be offering a *sentential* stipulation, of '$NxFx = NxGx$'.
- Let's take them in turn.

Subsentential stipulation

- If we offer HP as a subsentential stipulation of '$Nx\ldots x\ldots$', we are stipulating the following:

  > '$Nx\ldots x\ldots$' is to refer to whatever function it needs to refer to for HP to be true.

- This sort of stipulation does fix the meaning of '$Nx\ldots x\ldots$'. In particular, it provides a criterion for deciding of a given function whether it is the function that '$Nx\ldots x\ldots$' picks out.

Subsentential stipulation

- Consider the earlier ‘Jack the Ripper’ example:

  > ‘Jack the Ripper’ is to refer to the person who committed the Whitechapel murders.

- This stipulation does give us a criterion for deciding of a given person whether they are the person referred to by ‘Jack the Ripper’: they are if they committed the Whitechapel murders.

Guaranteed truth?

- But neo-Fregeans also want their stipulation to guarantee the truth of HP.
- Here, subsentential stipulation seems hopeless.
- Stipulating that ‘Jack the Ripper’ refers to the person who committed the Whitechapel murders does not guarantee that ‘Jack the Ripper committed the Whitechapel murders’ is true.
- Perhaps no one committed them, or perhaps many people did.
- What the stipulation guarantees is that the conditional

  > If someone committed the Whitechapel murders, then Jack the Ripper did.

  is true.

Carnap sentences

- In David Lewis's famous 1970 'How to define theoretical terms', he introduces the notion of a *Carnap sentence*, which is useful here.
- Consider some theory, which we can write as a sentence which is the conjunction of its axioms 'T'. 'T' contains the theoretical term 't', so we'll write the sentence as '$T(t)$'.
- We can define 't' by stipulating that 't' refers to the thing that satisfies '$T(x)$'.
- This stipulation does not guarantee that '$T(t)$' is true. Rather it guarantees the truth of T's Carnap sentence:

  $$\exists x Tx \rightarrow Tt$$

Carnap sentences

- Consider a theory with just the axiom 'Jack the Ripper committed the Whitechapel murders'.
- If we want that theory to implicitly define 'Jack the Ripper', then all we guarantee the truth of is the Carnap sentence of that theory.
- Similarly, the *subsentential* stipulation of '$Nx...x...$' does not guarantee the truth of HP but only of the Carnap sentence:

$$\exists f \forall F \forall G(fxFx = fxGx \leftrightarrow F \sim G) \rightarrow$$
$$\forall F \forall G(NxFx = NxGx \leftrightarrow F \sim G)$$

Sentential stipulation

- Perhaps, then, the neo-Fregean should offer a *sentential* stipulation of HP:

  > '$NxFx = NxGx$' is to have the same truth-value as '$F \sim G$' on every assignment of values to 'F' and 'G'.

- This stipulation obviously guarantees the truth of HP.
- It stipulates that the two halves of HP have the same truth-value and, since HP is a biconditional, that serves to guarantee that HP is true.
- The problem this time is fixing a meaning for the term-forming operator '$Nx...x...$'.

Complexity

- The problem is that the sentential stipulation tells us nothing at all about the complexity of '$NxFx = NxGx$'.
- All we are told is that instances of '$NxFx = NxGx$' have the same truth-value as corresponding instances of '$F \sim G$'.
- All we know, then, is that the syntactic unit '$NxFx = NxGx$' is an unstructured open sentence with 'F' and 'G' free.

An analogy

- Suppose I introduce a sentence '$F(Socrates)$' and fix its truth-value by stipulating that it is true.
- Are we allowed, on the basis of this stipulation, to take 'Socrates' as a singular term?
- It seems not, since we should then be allowed to substitute it for other names, but the stipulation is silent on the truth-value of '$F(Plato)$'.

An analogy

- Now suppose I introduce to the sentence '$G(Socrates)$' by stipulating that 'G' is true of an object iff that object committed suicide.
- The truth-value of '$G(Socrates)$' is in part fixed by the referent of 'Socrates'.
- But the truth-value of '$F(Socrates)$' is not. In '$F(Socrates)$', 'Socrates' is not a relevant semantic unit.

Identity

- The usual response from neo-Fregeans is to insist that the sentence '$NxFx = NxGx$' *does* have some structure: '$=$' is the identity sign, and the identity sign must be flanked by singular terms, so '$NxFx$' and '$NxGx$' *are* terms.
- But the neo-Fregean is not allowed to make this response. When we fix the truth-value of a sentence, the way in which we do so settled the role that the parts of that sentence play in determining meaning.
- If we do not treat '$NxFx$' as a semantic unit when fixing the meaning, then it plays no role in fixing the truth-value of the whole.
- And when we sententially stipulate HP, we are not treating '$NxFx$' as a semantic unit: we are treating '$NxFx = NxGx$' as a semantic unit, and have no right to isolate any part of it.

Conclusion

- Neo-logicists believe that Frege went wrong in abandoning HP as a definition of number.
- They come in two broad flavours: analytic logicists and neo-Fregan logicists.
- The former attempt to reduce arithmetic to the analytic but face problems with the ontological commitments of second-order logic and HP.
- The latter attempt to stipulate HP as an implicit definition of number. This has clear semantic and epistemological advantages.
- But they must either subsententially or sententially stipulate HP and both ways face problems. The first fixes a meaning for the term-forming operator but does not guarantee the truth of HP. The latter guarantees the truth but fails to fix the meaning.