Philosophy of Mathematics
Nominalism

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Structuralism

- Last week, we looked at *structuralism* as a philosophy of mathematics.
- We focussed on a Shapiro's *platonist* structuralism.
- This week, we'll consider two sorts of *nominalism*.
- The first is Hellman's nominalist structuralism.
- The second is Field's fictionalism.

Nominalist structuralism

- Given the problems faced by platonist structuralists, Hellman proposes a *modal* structuralism.
- The central idea is that we can enjoy the benefits of platonist structuralism but without the ontological commitment.
- Rather than thinking about *actual* structures, we need only be concerned with *possible* structures.

Modal structuralism

- Consider some arithmetic sentence S. Hellman would paraphrase S:

  $$\Box\forall X(X \text{ is an } \omega\text{-sequence} \rightarrow S \text{ holds in } X)$$

- In words: necessarily, if there is an ω-sequence, then S is true in that sequence.
- Hellman calls this the *hypothetical component* of his view.

Vacuity

- This paraphrase faces an immediate vacuity problem.
- The nominalist structuralist does not want to be committed to mathematical objects, since they are abstract and so present e.g. access problems.
- These are the very things that caused issues for Shapiro.
- But, for all we know, the universe may not contain enough physical objects for all of mathematics.
- E.g. if the universe contains only $10^{100,000}$ objects, then there won't be any ω-sequences.

Vacuity

- Even if we are convinced that the universe contains enough objects for arithmetic, there may be some limit to the number of objects it contains, and there will be branches of mathematics that require more.
- If there are not enough objects, then the antecedent of the hypothetical component will be false and so the whole conditional will be vacuously true.
- For this reason, Hellman introduces the categorical component:

  $$\Diamond\exists X(X \text{ is an } \omega\text{-sequence})$$

- In words: it is possible that an ω-sequence exists.

The categorical component

- The categorical component guarantees that, if the hypothetical component is true, it is non-vacuously true.
- There's one wrinkle. Both components are expressed in second-order S5. But it had better be S5 without the *Barcan Formula*:

  BF $\Diamond\exists X\phi \rightarrow \exists X\Diamond\phi$

- With BF, the categorical component entails:

  $$\exists X\Diamond(X \text{ is an } \omega\text{-sequence})$$

  which Hellman obviously doesn't want.

Modality

- Overall, then, the thought is that mathematical sentences are elliptical for longer sentences in second-order S5 (without BF).
- They have a hypothetical component to achieve this, and a categorical component to avoid vacuity.
- The account therefore avoids Shapiro's metaphysical burden
- But it loses Shapiro's ability to take mathematical sentences at face value.

Modality

- Further, the nature of the invoked modality must also be explained.
- Is it metaphysical possibility, or logical, or mathematical?
- Hellman says logical, but now there's a problem.
- Logical modality usually gets explicated in set-theoretic terms, but that is to let abstract objects back in.
- Instead, he refuses to explicate the logical modality at all, but leaves it primitive.

Ontology vs ideology

- Quine says that a theory carries an *ontology* and an *ideology*
- The ontology consists of the entities which the theory says exist.
- The ideology consists of the ideas expressed within the theory using predicates, operators, etc.
- Ontology is measured by the number of entities postulated by a theory.
- Ideology is measured by the number of primitives.
- It is often thought that ideological economy has epistemological benefits. A theory with fewer primitives is likely to be more unified, which may aid understanding.

Nominalism

- In summary, Shapiro can read mathematical sentences at face value but, as a result, gets the increased ontology.
- Hellman avoids the ontological commitment, but loses the face-value readings.
- We could try to combine the attractive features of each: mathematical sentences are taken at face value, so number terms are singular terms.
- But they are *empty* singular terms, since there are no numbers.
- As a result, mathematical sentences are, strictly, false.
- The obvious problem is: if maths is false, how is it so useful?

The indispensability argument

- The indispensability argument was put forward by Quine and Putnam as an argument for platonism about mathematical objects.
- Put simply, it has the following form:
  1. We ought to be ontologically committed to all and only those entities that are indispensable to our best scientific theories.
  2. Mathematical entities are indispensable to our best scientific theories.

  ∴ 3 We ought to be ontologically committed to mathematical entities.

Fictionalism

- Harty Field, in his *Science Without Numbers* (1980) argues that mathematical objects such as numbers do not exist.
- As such, singular terms that purport to refer to numbers are in fact empty terms and so mathematical sentences are false.
- Field's first task, therefore, is to undermine the indispensability argument.
- He argues that, in fact, mathematics *is* dispensable to science because mathematized scientific theories are *conservative* over their nominalised counterparts.

Conservativeness

- A theory Θ_2 is a *conservative extension* of a theory Θ_1 iff
  1. the language of Θ_2 extends the language of Θ_1;
  2. every theorem of Θ_1 is a theorem of Θ_2; and
  3. any theorem of Θ_2 that is not a theorem of Θ_1 is not expressible in the language of Θ_1.
- As it stands, this definition is ambiguous between semantic and syntactic notions of 'theorem'.

Nominalism and conservativeness

- Let M be a mathematized theory, and N be its nominalistic counterpart.
- Where M contains mathematical terms, N contains nominalistically respectable counterparts.
- M is a conservative extension of N iff
  1. the language of M extends that of N;
  2. every theorem of N is a theorem of M; and
  3. any theorem of M that is not a theorem of N is not expressible in the language of N.

Nominalism and conservativeness

- Field claims that mathematized theories conservatively extend their nominalistic counterparts.
- We could restrict ourselves to the nominalistic theories, and we wouldn't be missing anything.
- But mathematics is still *useful*, e.g. it can facilitate inference more quickly.

Nominalism and conservativeness

- For example, let A be some nominalistic sentence that follows from nominalistic sentences $N_1, ..., N_n$.
- We could derive A from $N_1, ..., N_n$ directly, but the derivation be long and time-consuming.
- In many cases, it will be easier to ascend to the mathematical counterparts $M_1^*, ..., M_n^*$, derive the mathematical A^* and finally descend to the nominalistic A.

Field's project

- Field's project has 2 parts: (i) produce the nominalistic theories that are the counterparts to mathematized theories; (ii) prove that the mathematical theories are conservative over their nominalistic counterparts.
- Of course, (i) is a huge task, since different nominalistic theories will be required for different mathematical theories. Field intends to instigate a research programme, but begins with an example.
- Field's example is Newtonian gravitational theory. He developed some techniques for nominalising it that can be used more widely.

Newtonian gravitational theory

- The basic ontology of Newtonian gravitational theory consists of ordered quadruples of real numbers and sets of real numbers.
- The vocabulary includes functors whose values are numbers, e.g. 'the gravitational potential of x'.
- Field replaces the quadruples with those of space-time points.
- The quadruples of real numbers are used to specify spatial coordinates in four-dimensional space. Field instead uses regions formed by space-time points.
- Instead of functors such as 'gravitational potential of x', Field uses comparative predicates, e.g. 'the difference in gravitational potential between x and y is less than that between z and w'.
- When completed, this has as semantic value a truth-value rather than a number.

Newtonian gravitational theory

- Field also needs *bridge principles* to move between the two.
- He calls these 'representation theorems'.
- A simple example connects numerical claims about distance with comparative claims about points.
- E.g. his nominalised theory contains the predicates

Bet(xyz)	x is a point on a line segment whose end points are y and z
Cong($xyzw$)	the line segment with end points x and y is congruent to the line segment with end points z and w

Representation theorems

- Field proves that there is a *distance* function d that maps pairs of space-time points to real numbers:
  1. for any points x, y, z and w, Cong($xyzw$) iff $d(x, y) = d(z, w)$
  2. for any points x, y and z, Bet(yxz) iff $d(x, y) + d(y, z) = d(x, z)$
- If d represents distance, then this theorem shows that claims about congruence and between-ness are equivalent to claims about distance.
- These allow us to pass from comparative claims about space-time points to abstract numerical claims about distances, and *vice versa*.

Representation theorems

- Field (pp. 61–91) provides other representation theorems.
- E.g. there is a 'spatio-temporal co-ordinate function' mapping space-time points into quadruples of reals.
- And there are 'gravitational potential' and 'mass density' functions mapping space-time points to intervals of reals.
- So abstract claims about gravitational potential and mass density are equivalent to comparative claims about space-time points.

Initial worries

- Some initial concerns:
  1. Field presupposes *substantivalism* about space-time: the view that space-time exists as a concrete entity. A controversial view.
  2. Field has chosen a scientific theory that is peculiarly amenable to nominalisation. It is generally thought that e.g. quantum mechanics would be much more challenging.
  3. He hasn't even come close to establishing that mathematics is dispensable to *all* scientific theories.
  4. Field's nominalised theory is second-order so, if we have worries about the ontological commitments of second-order logic, they will also apply here.

Conservativeness

($\models$) M is a conservative extension of N iff

1. the language of M extends that of N;
2. every sentence that is a **logical consequence** of N is a **logical consequence** of M; and
3. any sentence that is a **logical consequence** of M that is not a **logical consequence** of N is not expressible in the language of N.

($\vdash$) M is a conservative extension of N iff

$1'$ the language of M extends that of N;

$2'$ every sentence **provable** in N is **provable** in M; and

$3'$ any sentence **provable** in M that is not **provable** in N is not expressible in the language of N.

Semantic conservativeness

- The semantic notion invokes *logical consequence*, which is defined in terms of models in the usual post-Tarskian way:

  > ϕ is a logical consequence of Γ iff every model of Γ is a model of ϕ

- Models are generally understood as ordered pairs of domain and valuation function.
- But domains are sets, which are mathematical objects.
- In this way, abstract entities have reappeared in the metalogic.

Syntactic conservativeness

- Syntactic conservativeness isn't as *obviously* mathematical.
- But it invokes *provability*:

  ϕ is provable from Γ iff there is proof of ϕ from Γ using only the rules of a standard Natural Deduction system

- This definition invokes 'proofs', but what are these? Proof *tokens* or *types*.
- The former are concretely inscibed proofs. But there aren't enough of them: there are many proofs which have never been written down.
- The latter are abstract types. Here we have all the proofs we need, but they are abstract once again.

Consistency

- At some points, Field suggests defining taking the semantic notion but defining it in terms of *consistency*:

  > ϕ is a logical consequence of Γ iff Γ anf $\neg\phi$ are logically inconsistent

- We now have a notion of logical consistency to explain, and the usual explanation is modal:

  > some sentences are logically consistent iff it is possible that they can all be true together

- We now have a modal notion to deal with. As with Hellman, what sort of modality is it?
- We could take the modality as primitive but, again like Hellman, that increases ideology.

Conclusion

- Field's fictionalism involves (i) nominalising scientific theories and (ii) proving that the mathematized theories are conservative over their nominalised counterparts.
- We have seen that there are problems with both parts.
- The nominalisation itself faces problems because it assumes substantivalism about space-time, considers only a relatively simple theory to nominalise, and relies on second-order logic.
- The conservativeness suffers because it must either be syntactic or semantic, and neither is available to a nominalist.
- Generally, as with Hellman's nominalism, Field *may* have succeeded in eliminating abstract entities from his mathematical theories, but has not succeeded in eliminating them from the metalogic he requires.