Consider the Pythagorean argument that $\sqrt{2}$ is irrational:

1. Assume that $\sqrt{2}$ is rational, so $\sqrt{2} = \frac{m}{n}$, where $m$ and $n$ are coprime.
2. Then $2n^2 = m^2$, so $m^2$ is even, so $m$ is even.
3. Then $m = 2k$ for some integer $k$. Substituting, $n^2 = 2k^2$, so $n^2$ is even, so $n$ is even.
4. Contradiction: if $m$ and $n$ are both even, then they are not coprime.
5. By reductio, $\sqrt{2}$ is irrational.
Hilbert’s 7th Problem: are there irrational numbers $\alpha$ and $\beta$ such that $\alpha^\beta$ is rational?

Here’s a classic answer:

1. $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational.
2. Assume it is rational. Then $\alpha = \beta = \sqrt{2}$ is a solution.
3. Assume it is irrational. Then a solution is $\alpha = \sqrt{2}^{\sqrt{2}}$ and $\beta = \sqrt{2}$, since $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$, which is rational.
4. Either way, there are such numbers.

This proof establishes that either $\langle \alpha, \beta \rangle = \langle \sqrt{2}, \sqrt{2} \rangle$ or $\langle \alpha, \beta \rangle = \langle \sqrt{2}^{\sqrt{2}}, \sqrt{2} \rangle$ is the solution. But it does not establish which is the correct disjunct.
Law of Excluded Middle

- Some mathematicians – intuitionists – would reject this proof, since no specific solution has been given.
- In particular, the proof used the Law of Excluded Middle:
  \[ \text{LEM} \vdash \phi \lor \neg\phi \]
- We used LEM at the fist stage, when we assumed that \( \sqrt{2} \) is either rational or irrational.
- LEM is the syntactic counterpart of the semantic Principle of Bivalence:
  \[ \text{PB} \models \phi \lor \neg\phi \]
Intuitionist mathematics

- Intuitionists tie mathematical truth and falsity to proof and disproof.
- A mathematical claim is true if there is a proof of it, and false if there is a proof of its absurdity.
- There are mathematical claims for which we have neither, hence the law of excluded middle is inappropriate.

  GC Every even integer greater than 2 is the sum of two primes.
This is in stark contrast to classical mathematics. The classical mathematician accepts LEM. Although there are claims that have not yet been proved or disproved, they are still either true or false, depending on mathematical reality. Classical mathematics is generally \textit{realist}. Intuitionists deny that there is a mind-independent mathematical reality. Intuitionist mathematics is generally \textit{antirealist}. 
All this may put you in mind of Kant, from whom intuitionists take the term ‘intuition’.

Intuitionists generally believe that, since there is no mind-independent mathematical reality, mathematics is based on a particular faculty of the mind.

It is a constructivist view, since mathematics is something made by an act of will rather than discovered. Mathematics is constructed or built.

It is an idealist view, since it bases mathematics on the nature of mind.

Kant was certainly an idealist. It is less clear if he was a constructivist. He uses the term ‘construction’, but it is not obvious that he meant it in this sense.
Brouwer’s intuitionism was the most important idealist philosophy of the 20th Century. Like Kant, he thought that arithmetic was synthetic *a priori*. Unlike Kant, he thought that geometry was analytic *a priori*. This is the reverse of Frege’s position. This is largely because of the development of non-Euclidean geometries in the 19th Century. He believes that mathematical constructions are mental entities. By ignoring this point, and allowing the use of LEM, much of classical mathematics had gone awry. Arithmetic and analysis is based on the ‘primordial intuition’ of time.
Brouwer did not formalise his intuitionistic theory.

This task was first undertaken by his student Arend Heyting.

Proof-theoretically, intuitionist logic is just like the proof system from forallx, but without TND.
Model-theoretically, intuitionists generally accept the BHK (Brouwer-Heyting-Kolmogorov) interpretation:

1. A proof of $A \rightarrow B$ is a proof which transforms any proof of $A$ into a proof of $B$.
2. A proof of $\neg A$ is a proof which transforms any hypothetical proof of $A$ into a proof of a contradiction.
3. A proof of $\forall x A(x)$ is a proof which transforms a proof of $d$ in $D$ (where $D$ is the domain over which the quantifier ranges) into a proof of $A(d)$.

Intuitionist mathematical theories are built on this logic.
Arithmetic

- The natural numbers are denumerably infinite.
- So intuitionist arithmetic (Heyting Arithmetic) largely agrees with classical arithmetic.
- The major difference is the background logic.
- The disagreements become apparent when we consider the real numbers.
Continuous functions

- A *continuous function* is one for which small changes in the input result in small changes in the output.
- Graphically, the graph contains no gaps.
A *discontinuous function* is one that is not continuous.

Classically, there are many discontinuous functions.
The continuum

- Intuitionistically, every function is continuous.
- We will not prove this, but give an intuitive gloss.
- Real numbers are infinite objects (unlike natural numbers).
- But the intuitionist does not accept the notion of a completed infinity.
- As such, intuitionists use finite approximations for real functions, e.g. the value of $f(x)$ up to $n$ decimal places.
The continuum

- For classically discontinuous functions, one cannot approximate the value at the point of discontinuity $x$ by considering $f(y)$ for any $y$ sufficiently close to $x$.
- Consider the function:
  
  $f(x) = 0$ if $x \neq 0$
  
  $f(x) = 1$ if $x = 0$

- No amount of finite information will determine of all possible arguments whether its value is arbitrarily close to 0 or 1.
- For any finite $n$, knowing that the first $n$ decimal places of $x$ are 0.0...0 leaves it open whether the value of $f(x)$ is close to 0 or 1.
- For at least one argument $x$, we need *infinite* information about $x$ to determine its value up to some desired level of precision.
Limitations

- As such, classically discontinuous function are intuitionistically banished.
- This is a significant weakening of classical analysis.
- The intuitionistic functions are a small subset of classical functions.
- Undurprisingly, intuitionistic set theory is also impoverished.
That’s an outline of intuitionistic mathematics, the logic on which it is built, and its philosophical underpinning.

Now let’s look at an argument for it.

The most famous modern defender of intuitionistic mathematics is Michael Dummett.

He has offered several influential arguments for intuitionistic mathematics, e.g. (i) manifestation argument; (ii) acquisition argument; (iii) argument from indefinite extensibility.

Let’s focus on (iii), since I lectured on (i) and (ii) in the \textit{Realism and Idealism} lectures. Also, it is the only distinctively mathematical argument: the others apply equally to all areas of discourse.
Classical quantification

- The classical interpretation of the first-order quantifiers is that they express infinitary truth functions.
- The functions are from a set of objects (finite or infinite) to a set of truth-values.
- A simple universally quantified sentence is true iff each of the objects in the domain is mapped to the value True.
- For classical quantification to be justified, therefore, it must be determinate which function is being expressed.
- This requires that it is *determinate* which objects belong to the domain of the function.
What is it for the domain to be *determinate*?

It is for the concepts involved in specifying the function to have determinate criteria of application and identity.

A determinate criterion for application is provided for concept $C$ just if, for any candidate object, it is determinate whether that object falls under $C$ or not.

A determinate criterion for identity is provided for concept $C$ just if, for any two objects falling under $C$, it is determinate whether they are identical or not.
Determinate concepts

- These criteria will give you conditions that must be satisfied for the concept to be determinate.
- If realism about these concepts is true, then reality will do the rest: it will determine which objects satisfy the conditions.
- When we have a definite concept and realism about that concept, classical quantification is justified.
Mathematical reality

- Classical quantification cannot be justified on the mathematical domain, according to Dummett.
- There is no hope of mathematical reality determining the appropriate domain of objects.
- It may be that there are determinate concepts of e.g. natural number, ordinal number and set.
- But mathematical reality cannot determine a definite totality of these objects to be the domain of quantification.
Suppose, for example, that we have a definite totality of the ordinal numbers.

If the totality is definite, then it is capable of being well-ordered, and so has an ordinal number.

But this ordinal must be greater than any in the totality.

So we did not start with a definite totality of ordinal numbers.
Dummett calls concepts such as set, ordinal number and cardinal number *indefinitely extensible* concepts.

Dummett’s views on indefinite extensibility change over time.

Here’s his notion in a late paper ‘What is mathematics about?’:

> An indefinitely extensible concept is one such that, if we can form a definite conception of a totality all of whose members fall under that concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it. Russell’s concept of class not a member of itself provides a beautiful example of an indefinitely extensible concept. (p. 441)
From indefinite extensibility to intuitionism

- It is easy to see how the existence of indefinitely extensible concepts may suggest that intuitionism is true.
- The realist will believe that mathematical reality is determinate and settles the truth-value of all mathematical sentences.
- But, when a concept is indefinitely extensible, it seems that mathematical reality is not there determinate.
- However we try to characterise the domain, we will miss something.
More precisely

1. The extensions of indefinitely extensible concepts do not constitute determinate domains of quantification.
2. ‘Set’ is an indefinitely extensible concept.
3. If a domain is not determinate, sentences quantifying over that domain need not have determinate truth-value.
4. Sentences quantifying over the domain of sets need not have determinate truth-value.
The first premise is true by the definition of ‘indefinite extensibility’.

We may doubt the second premise in some cases.

E.g. the intuitive conception of set is paradoxical but more sophisticated conceptions are not.

And other concepts are not obviously problematic in the same way, e.g. real number.

Dummett thinks, however, that there are more indefinitely extensible concepts than the paradoxical ones.
Premise 2

- A concept such as ‘natural number’ has an *intrisically infinite* extension: we ‘have means of finding another element of the totality, however many we have already identified’ (p. 318).
- Suppose we have a totality of natural numbers, ending with $n$.
- Then we can specify a new one, $n + 1$.
- The ‘means’ of finding a new object is the possession of a principle of extensibility, in this case the successor function.
- The concept of natural number is not thought to be paradoxical.
- It is, nevertheless, indefinitely extensible.
The crucial premise is 3.

3 If a domain is not determinate, sentences quantifying over that domain need not have a truth-value.

We may deny this by claiming that we do in fact possess a conception of a definite totality of natural numbers sufficient to justify classical quantification.
Premise 3

- Dummett denies this:
  
  *No refutation can be devised to defeat, on his own ground, a finitist who professes not to understand the conception of any infinite totality: Frege was mistaken in supposing that there can be a proof that such a totality exists which must convince anyone capable of reasoning* (p. 234)

- The worry is a sceptical one: how do we know that we have the correct conception?

- Even if we have a story to back up our conception, it must – for Dummett – be communicable and graspable.

- But demands of communicability and graspability seem to push us back to the usual meaning-theoretic arguments.
Conclusions

- We’ve now seen 4 of the most important contemporary philosophies of mathematics: neo-logicism, structuralism, fictionalism and intuitionism, and their historical inspirations.
- Neo-logicism, inspired by Fregean logicism, faces problems relating to the notion of analytiicity or the implicit definition involved.
- Structuralism, inspired by Kant, faces epistemological problems (when platonist) and ideological problems (when nominalist).
- Fictionalism is limited, and faces indispensability worries.
- Intuitionism severely limits mathematics, and Dummett’s arguments may lead us back to the old manifestation and acquisition arguments.