Philosophy of Mathematics
Intuitionism

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Classical mathematics

- Consider the Pythagorean argument that $\sqrt{2}$ is irrational:
  1. Assume that $\sqrt{2}$ is rational, so $\sqrt{2} = \frac{m}{n}$, where m and n are *coprime*.
  2. Then $2n^2 = m^2$, so m^2 is even, so m is even.
  3. Then $m = 2k$ for some integer k. Substituting, $n^2 = 2k^2$, so n^2 is even, so n is even.
  4. Contradiction: if m and n are both even, then they are not coprime.
  5. By *reductio*, $\sqrt{2}$ is irrational.

Classical mathematics

- Hilbert's 7th Problem: are there irrational numbers α and β such that α^β is rational?
- Here's a classic answer:
  1. $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational.
  2. Assume it is rational. Then $\alpha = \beta = \sqrt{2}$ is a solution.
  3. Assume it is irrational. Then a solution is $\alpha = \sqrt{2}^{\sqrt{2}}$ and $\beta = \sqrt{2}$, since $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$, which is rational.
  4. Either way, there are such numbers.
- This proof establishes that either $\langle \alpha, \beta \rangle = \langle \sqrt{2}, \sqrt{2} \rangle$ or $\langle \alpha, \beta \rangle = \langle \sqrt{2}^{\sqrt{2}}, \sqrt{2} \rangle$ is the solution. But it does not establish which is the correct disjunct.

Law of Excluded Middle

- Some mathematicians – intuitionists – would reject this proof, since no specific solution has been given.
- In particular, the proof used the Law of Excluded Middle:

  LEM $\vdash \phi \vee \neg\phi$

- We used LEM at the fist stage, when we assumed that $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational.
- LEM is the syntactic counterpart of the semantic *Principle of Bivalence*:

  PB $\models \phi \vee \neg\phi$

Intuitionist mathematics

- Intuitionists tie mathematical truth and falsity to proof and disproof.
- A mathematical claim is true if there is a proof of it, and false if there is a proof of its absurdity.
- There are mathematical claims for which we have neither, hence the law of excluded middle is inappropriate.

  GC Every even integer greater than 2 is the sum of two primes.

Intuitionism and antirealism

- This is in stark contrast to classical mathematics.
- The classical mathematician accepts LEM.
- Although there are claims that have not yet been proved or disproved, they are still either true or false, depending on mathematical reality.
- Classical mathematics is generally *realist*.
- Intuitionists deny that there is a mind-independent mathematical reality.
- Intuitionist mathematics is generally *antirealist*.

Kant

- All this may put you in mind of Kant, from whom intuitionists take the term ‘intuition’.
- Intuitionists generally believe that, since there is no mind-independent mathematical reality, mathematics is based on a particular faculty of the mind.
- It is a *constructivist* view, since mathematics is something made by an act of will rather than discovered. Mathematics is *constructed* or *built*.
- It is an *idealist* view, since it bases mathematics on the nature of mind.
- Kant was certainly an idealist. It is less clear if he was a constructivist. He uses the term ‘construction’, but it is not obvious that he meant it in this sense.

Brouwer

- Brouwer's intuitionism was the most important idealist philosophy of the 20th Century.
- Like Kant, he thought that arithmetic was synthetic *a priori*.
- Unlike Kant, he thought that geometry was analytic *a priori*.
- This is the reverse of Frege's position.
- This is largely because of the development of non-Euclidean geometries in the 19th Century.
- He believes that mathematical constructions are mental entities.
- By ignoring this point, and allowing the use of LEM, much of classical mathematics had gone awry.
- Arithmetic and analysis is based on the 'primordial intuition' of time.

Intuitionistic logic: proof theory

- Brouwer did not formalise his intuitionistic theory.
- This task was first undertaken by his student Arend Heyting.
- Proof-theoretically, intuitionist logic is just like the proof system from *forallx*, but without TND.

Intuitionistic logic: model theory

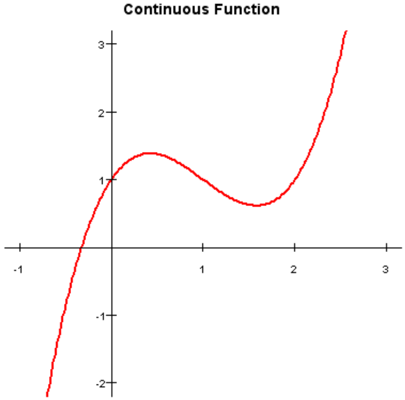
- Model-theoretically, intuitionists generally accept the BHK (Brouwer-Heyting-Kolmogorov) interpretation:
  1. A proof of $A \to B$ is a proof which transforms any proof of A into a proof of B.
  2. A proof of $\neg A$ is a proof which transforms any hypothetical proof of A into a proof of a contradiction.
  3. A proof of $\forall x A(x)$ is a proof which transforms a proof of d in D (where D is the domain over which the quantifier ranges) into a proof of $A(d)$.
- Intuitionist mathematical theories are built on this logic.

Arithmetic

- The natural numbers are denumerably infinite.
- So intuitionist arithmetic (Heyting Arithmetic) largely agrees with classical arithmetic.
- The major difference is the background logic.
- The disagreements become apparent when we consider the real numbers.

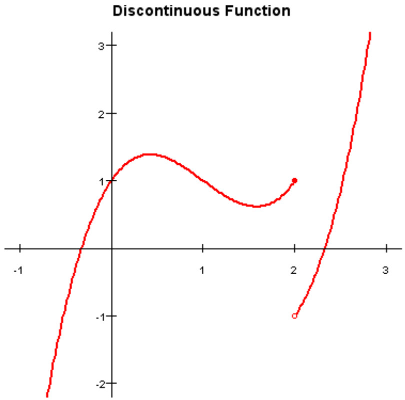
Continuous functions

- A *continuous function* is one for which small changes in the input result in small changes in the output.
- Graphically, the graph contains no gaps.



Dicontinuous functions

- A *discontinuous function* is one that is not continuous.
- Classically, there are many discontinuous functions.



The continuum

- Intuitionistically, every function is continuous.
- We will not prove this, but give an intuitive gloss.
- Real numbers are infinite objects (unlike natural numbers).
- But the intuitionist does not accept the notion of a completed infinity.
- As such, intuitionists use finite approximations for real functions, e.g. the value of $f(x)$ up to n decimal places.

The continuum

- For classically discontinuous functions, one cannot approximate the value at the point of discontinuity x by considering $f(y)$ for any y sufficiently close to x.
- Consider the function:

  $f(x) = 0$ if $x \neq 0$
  $f(x) = 1$ if $x = 0$
- No amount of finite information will determine of all possible arguments whether its value is arbitrarily close to 0 or 1.
- For any finite n, knowing that the first n decimal places of x are 0.0...0 leaves it open whether the value of $f(x)$ is close to 0 or 1.
- For at least one argument x, we need *infinite* information about x to determine its value up to some desired level of precision.

Limitations

- As such, classically discontinuous function are intuitionistically banished.
- This is a significant weakening of classical analysis.
- The intuitionistic functions are a small subset of classical functions.
- Undurprisingly, intuitionistic set theory is also impoverished.

Michael Dummett

- That's an outline of intuitionistic mathematics, the logic on which it is built, and its philosophical underpinning.
- Now let's look at an argument for it.
- The most famous modern defender of intuitionistic mathematics is Michael Dummett.
- He has offered several influential arguments for intuitionistic mathematics, e.g. (i) manifestation argument; (ii) acquisition argument; (iii) argument from indefinite extensibility.
- Let's focus on (iii), since I lectured on (i) and (ii) in the *Realism and Idealism* lectures. Also, it is the only distinctively *mathematical* argument: the others apply equally to all areas of discourse.

Classical quantification

- The classical interpretation of the first-order quantifiers is that they express infinitary truth functions.
- The functions are from a set of objects (finite or infinite) to a set of truth-values.
- A simple universally quantified sentence is true iff each of the objects in the domain is mapped to the value True.
- For classical quantification to be justified, therefore, it must be determinate which function is being expressed.
- This requires that it is *determinate* which objects belong to the domain of the function.

Determinate concepts

- What is it for the domain to be *determinate*?
- It is for the concepts involved in specifying the function to have determinate criteria of application and identity.
- A determinate criterion for application is provided for concept C just if, for any candidate object, it is determinate whether that object falls under C or not.
- A determinate criterion for identity is provided for concept C just if, for any two objects falling under C, it is determinate whether they are identical or not.

Determinate concepts

- These criteria will give you conditions that must be satisfied for the concept to be determinate.
- If realism about these concepts is true, then reality will do the rest: it will determine which objects satisfy the conditions.
- When we have a definite concept and realism about that concept, classical quantification is justified.

Mathematical reality

- Classical quantification cannot be justified on the mathematical domain, according to Dummett.
- There is no hope of mathematical reality determining the appropriate domain of objects.
- It may be that there are determinate concepts of e.g. natural number, ordinal number and set.
- But mathematical reality cannot determine a definite totality of these objects to be the domain of quantification.

Ordinals

- Suppose, for example, that we have a definite totality of the ordinal numbers.
- If the totality is definite, then it is capable of being well-ordered, and so has an ordinal number.
- But this ordinal must be greater than any in the totality.
- So we did not start with a definite totality of ordinal numbers.

Indefinite extensibility

- Dummett calls concepts such as set, ordinal number and cardinal number *indefinitely extensible* concepts.
- Dummett's views on indefinite extensibility change over time.
- Here's his notion in a late paper 'What is mathematics about?':

  > *An indefinitely extensible concept is one such that, if we can form a definite conception of a totality all of whose members fall under that concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it. Russell's concept of class not a member of itself provides a beautiful example of an indefinitely extensible concept. (p. 441)*

From indefinite extensibility to intuitionism

- It is easy to see how the existence of indefinitely extensible concepts may suggest that intuitionism is true.
- The realist will believe that mathematical reality is determinate and settles the truth-value of all mathematical sentences.
- But, when a concept is indefinitely extensible, it seems that mathematical reality is not there determinate.
- However we try to characterise the domain, we will miss something.

More precisely

1 The extensions of indefinitely extensible concepts do not constitute determinate domains of quantification.

2 ‘Set’ is an indefinitely extensible concept.

3 If a domain is not determinate, sentences quantifying over that domain need not have determinate truth-value.

∴ 4 Sentences quantifying over the domain of sets need not have determinate truth-value.

Premise 2

- The first premise is true by the definition of ‘indefinite extensibility’.
- We may doubt the second premise in some cases.
- E.g. the intuitive conception of set is paradoxical but more sophisticated conceptions are not.
- And other concepts are not obviously problematic in the same way, e.g. real number.
- Dummett thinks, however, that there are more indefinitely extensible concepts than the paradoxical ones.

Premise 2

- A concept such as ‘natural number’ has an *intrisically infinite* extension: we ‘have means of finding another element of the totality, however many we have already identified’ (p. 318).
- Suppose we have a totality of natural numbers, ending with n.
- Then we can specify a new one, $n + 1$.
- The ‘means’ of finding a new object is the possession of a principle of extensibility, in this case the successor function.
- The concept of natural number is not thought to be paradoxical.
- It is, nevertheless, indefinitely extensible.

Premise 3

- The crucial premise is 3.

3 If a domain is not determinate, sentences quantifying over that domain need not have a truth-value.

- We may deny this by claiming that we *do* in fact possess a conception of a definite totality of natural numbers sufficient to justify classical quantification.

Premise 3

- Dummett denies this:

  > *No refutation can be devised to defeat, on his own ground, a finitist who professes not to understand the conception of any infinite totality: Frege was mistaken in supposing that there can be a proof that such a totality exists which must convince anyone capable of reasoning (p. 234)*

- The worry is a sceptical one: how do we know that we have the correct conception?
- Even if we have a story to back up our conception, it must – for Dummett – be communicable and graspable.
- But demands of communicability and graspability seem to puch us back to the usual meaning-theoretic arguments.

Conclusions

- We've now seen 4 of the most important contemporary philosophies of mathematics: neo-logicism, structuralism, fictionalism and intuitionism, and their historical inspirations.
- Neo-logicism, inspired by Fregean logicism, faces problems relating to the notion of analytciity or the implicit definition involved.
- Structuralism, inspired by Kant, faces epistemological problems (when platonist) and ideological problems (when nominalist).
- Fictionalism is limited, and faces indispensability worries.
- Intuitionism severely limits mathematics, and Dummett's arguments may lead us back to the old manifestation and acquisition arguments.