

Philosophy of Mathematics

Kant

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Immanuel Kant

- ▶ Born in 1724 in Königsberg, Prussia.
- ▶ Enrolled at the University of Königsberg in 1740 and remained there his whole career.
- ▶ Published *The Critique of Pure Reason* in 1781.
- ▶ Was not a mathematician, but lectured on mathematics for 8 years in Königsberg.
- ▶ After his death, his library was found to contain many mathematics texts, including Newton's *Principia* and Euler's *Mechanica*.

The *a priori*/ *a posteriori* distinction

- ▶ An epistemological distinction.
- ▶ Kant applies it to *cognition*. I will apply it to *knowledge*.
- ▶ A truth is known *a priori* if it is known 'absolutely independently of all experience and even of all impressions of the senses' (B2–3).
- ▶ '[E]xperience teaches us ... that something is constituted thus and so, but not that it could not be otherwise' (B3).
- ▶ '[E]xperience never gives its judgements true or strict universality' (B3).
- ▶ *A priori* knowability entails necessity and universality/ generality.

The analytic/ synthetic distinction

- ▶ A semantic distinction introduced by Kant.
- ▶ Now, a standard definition is e.g.
 - A sentence is analytic iff it is true by definition.
 - A sentence is analytic iff it is a tautology, or reducible to one by substitution.
 - A sentence is analytic iff it is true in virtue of the meanings of the words.
- ▶ Such definitions face well-known problems from Quine.
- ▶ And these are not the sorts of definitions offered by Kant.

Kant's definition

- ▶ Kant applies it to *judgements*. I will apply it to *sentences*.
- ▶ A sentence in subject-predicate form is *analytic* 'if the predicate *B* belongs to the subject *A* as something that is (covertly) contained in the concept *A*' (A6/B10).
- ▶ Otherwise, it is *synthetic*.
- ▶ Analytic sentences add 'nothing through the predicate to the concept of the subject, but merely [break] it up into those constituent concepts that have been thought in it, although confusedly' (A7/B11).

Conceptual containment

- ▶ This definition relies on the obscure *conceptual containment* metaphor.
- ▶ Frege criticises the metaphor:

Kant obviously – as a result, no doubt, of defining them too narrowly – underestimated the value of analytic judgements ... [Mathematical concepts] are contained in their definitions, but as plants are contained in their seeds, not as beams are contained in a house. Often we need several definitions for the proof of some proposition, which consequently is not contained in any of them alone. (Grundlagen §88)
- ▶ And it only applies to sentences in subject-predicate form.
- ▶ Let's therefore look to another definition offered by Kant: the analytic as *explicative*.

The analytic as explicative

- ▶ Kant (A7/B11) characterised analyticity as being *explicative*, rather than *ampliative*.
- ▶ If we understand an analytic sentence, we learn nothing new when we come to know its truth.
- ▶ In this sense, analytic sentences cannot *amplify* our knowledge: they are merely *explicative*.

Independence of objects

- ▶ Kant gives us two tests for analyticity in this sense.
- ▶ First, the truth of a synthetic sentence requires something else for its truth.
- ▶ Because they are ampliative, synthetic sentences depend on the existence of at least one object.
- ▶ The truth of an analytic sentence is independent of all objects whatsoever.

Law of contradiction

- ▶ Second, analytic sentences 'can always be adequately known in accordance with the principle of contradiction' (A150–153/B189–192).
- ▶ If the negation of a sentence S is a contradiction, then S is analytic.
- ▶ All truths must accord with the law of non-contradiction, but the law provides a *sufficient* condition for analyticity.
- ▶ So logical truths are analytic, since the negation of a logical truth is a contradiction. But not all analytic sentences are logical truths.
- ▶ E.g. 'Haters gonna hate' is plausibly explicative, and so analytic, but is not a logical truth.

Law of contradiction

- ▶ '7+5≠ 13' is a mathematical truth.
- ▶ Is its negation, '7+5=13', a contradiction?

For thought a synthetic sentence can indeed be discerned in accordance with the principle of contradiction, this can only be if another synthetic sentence be presupposed; ... it can never be so discerned in and by itself (B14)

- ▶ A statement of identity (as opposed to non-identity) can only be a contradiction in the presence of other assumptions.
- ▶ It will only be the case that every mathematical truth has a contradictory negation in the presence of other assumptions, namely *axioms*.

Axioms

- ▶ Is the negation of an *axiom* analytic?
- ▶ No: the negation of an axiom is not a contradiction, since it would then be a logical truth!
- ▶ We can see already why Kant's notion of analyticity is not available to the logicist.
- ▶ Axioms must be deniable without contradiction.
- ▶ The law of contradiction gives us reason to think that axioms are synthetic.
- ▶ Any mathematical truth that depends essentially on an axiom is likewise synthetic.

Independence

- ▶ How about the other test: independence from objects?
- ▶ Taken at face-value, the axioms of arithmetic appear to require the existence of objects, e.g. a referent for '0'.
- ▶ We could paraphrase these away, or attempt to reduce them to logic.
- ▶ But, as we'll see, paraphrase projects still need objects, e.g. space-time points.
- ▶ And logicism, as we'll see, needs second-order logic, which has amongst its logical truths sentences like:

$$\exists X \forall x (Xx \leftrightarrow x \neq x)$$

$$\exists X \forall x (Xx \leftrightarrow x = x)$$

Generality

- ▶ Is this too stringent a condition? After all, first-order logic (classically understood) has $\exists x x \neq x$ as a logical truth.
- ▶ Kant holds that the non-ampliativity of logic is essential. This is what gives logic its distinctive *generality*.
- ▶ Logic should not be in the business, even partially, of answering the question, 'What is there?'.
- ▶ Thus Kant: I 'admit, as every reasonable person must, that all existential propositions are synthetic' (A598=B626).

Modern mathematics

- ▶ Does the development of mathematics since Kant support his claim that mathematics is synthetic?
- ▶ Hilbert's completion of the axiomatisation of Euclidean geometry, and Peano's of arithmetic, certainly appear synthetic in Kant's sense.
- ▶ Other areas, however, seem to pull against Kant. E.g. group theory doesn't seem to consist of *truths* in the same way as arithmetic and geometry.
- ▶ Many hold an *implicationist* view of group theory: that theorem ϕ follows from axioms Γ should be understood as 'if Γ , then ϕ ', rather than the categorical assertion of ϕ .

The axiomatic method

- ▶ So, for Kant, the analytic sentences are the explicative (non-ampliative) statements.
- ▶ Mathematical sentences are synthetic by this definition: they rely on axioms, whose negations are not contradictions, and require the existence of objects for their truth.
- ▶ This reading is slightly anachronistic. Kant did not share the modern notion of an axiom:

Certainly, arithmetic has no axioms, since its object is actually not any quantum, that is any quantitative object of intuition, but rather quantity as such, that is, it considers the concept of a thing in general by means of quantitative determination.

(Correspondence, p. 284)

The synthetic *a priori*

- ▶ Kant holds that mathematics consists of a body of synthetic *a priori* truths.
- ▶ A major part of the *Critique* is to argue that such truths are even possible.
- ▶ Why might we think there's a tension?
- ▶ Synthetic sentences, we have seen, answer to independent objects.
- ▶ But, if we must experience these independent objects in order to gain knowledge of the synthetic sentence, then that knowledge, it seems, must be *a posteriori*.

Kant's answer

- ▶ Kant resolves this tension with his extraordinary *Transcendental Deduction*.
- ▶ This is an argument for his *transcendental idealism*.
- ▶ To make sense of this position, we must first introduce some of the basic elements of Kant's metaphysics.

Intuitions

- ▶ There are 2 fundamental components to our thoughts: *intuitions* and *concepts*.
- ▶ An intuition is an immediate representation of an object.
- ▶ Not to be confused with the usual English meaning of 'intuition'. For Kant, an intuition is more like an immediate perception.
- ▶ Intuitions are given to us by the faculty of *sensibility*.
- ▶ There is an analogy here with *logically proper names*.

Concepts

- ▶ A concept is a mediated representation of one object, many objects, or no objects.
- ▶ Concepts are given to us by the faculty of *understanding*.
- ▶ There is an analogy here with *predicates*.

The Copernican Turn

Up to now it has been assumed that all our cognition must conform to the objects. ... [Let] us once try whether we do not get farther with the problems of metaphysics by assuming that the objects must conform to our cognition. ... This would be just like the first thoughts of Copernicus, who, when he did not make good progress in the explanation of the celestial motions if he assumed that the entire celestial host revolves around the observer, tried to see if he might have greater success if he made the observer revolve around the stars at rest. (Bxvi)

The Transcendental Deduction

- ▶ We can gain knowledge of the external world by intuitions, which are given to us by the faculty of sensibility.
- ▶ Sensibility is essentially passive.
- ▶ The *structure* of the world is not given by sensibility, but imposed by us.
- ▶ It is as though we have native spectacles through which we filter the world.
- ▶ These spectacles cannot be removed.

Space and time

- ▶ In particular, our spectacles supply the spatial and temporal structure of the world.
- ▶ The spatial-temporal structure of the world is not independent of my internal faculties.
- ▶ Space and time are *projected* onto the world, giving it shape.
- ▶ Because space and time have their origin in the mind, they can be accessed *a priori*.
- ▶ This picture is in stark contrast to philosophies of space and time such as *four-dimensionalism*, and indeed to any modern metaphysics that claims to *carve reality at the joints*: we supply the joints!

Transcendental idealism

- ▶ The external world is mind-independent *but only partially*.
- ▶ The world is mind-independent, but the way we experience the world is mind-dependent.
- ▶ Kant's view therefore qualifies as *idealist*.
- ▶ In particular, it is a *transcendental idealism*.

Geometry

- ▶ The propositions of geometry are not about actual shapes but ideal ones.
- ▶ We may prove propositions using e.g. actual triangles, but must be careful to attend to its general properties: those it has in virtue of being a triangle.
- ▶ Our conclusions will then be about ideal triangles.
- ▶ Ideal triangles are products of the mind-dependent spatial structure of the world.
- ▶ Such knowledge is *a priori*.
- ▶ But it is also synthetic: it is not knowledge of definitions or tautologies.

Arithmetic

- ▶ Similarly, arithmetic knowledge is knowledge of the temporal and spatial structure of the world.
- ▶ We may exemplify e.g. the number 5 using fingers, but it is the structure that is important.
- ▶ The temporal aspect explains how arithmetic can be applied to non-physical entities.
- ▶ The infinity of arithmetic is explained by the infinite nature of space and time.
- ▶ Once again, this knowledge is synthetic and *a priori*.

Summing up

- ▶ For Kant, mathematics is a body of synthetic *a priori* truths.
- ▶ A *synthetic* truth is an ampliative one, the denial of which is not a contradiction, and which depends on the existence of objects.
- ▶ Mathematical truths are synthetic since their denial is not always a contradiction, their truth relies on the external world presented in intuition, and they are not trivial.
- ▶ Mathematical truths are *a priori* since they concern the temporal and spatial structure of reality. This structure is supplied by our minds, and we therefore have *a priori* access to it.
- ▶ The major critic of Kant is Gottlob Frege, whose *Grundlagen* we will consider next.