Philosophy of Mathematics
Kant and Frege

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Last week

- Kant argued that mathematical truths are synthetic and *a priori*.
- They are synthetic since they are non-trivial, ampliative, depend on the existence of objects and are deniable without contradiction.
- They are *a priori* knowable since they concern the spatio-temporal structure of reality. We have *a priori* access to this stucture, since it is supplied by us, rather than inherent in the world.

Gottlob Frege

- ▶ Born in 1848, enrolled at the University of Jena in 1869 and stayed their throughout his career.
- ▶ His doctoral thesis was on geometry and he was employed at Jena as a mathematician.
- ▶ In 1879, he published *Begriffschrift* (*Conceptual Notation*), which introduced an (idiosyncratic) formalization of first-order logic. This was arguably the greatest development in logic since Aristotle.
- ▶ In 1884, he published *Die Grundlagen der Arithmetik* (*The Foundations of Arithmetic*), which defends a radically non-Kantian position. It is arguably the most important text in the philosophy of mathematics.

Frege's aims

- ▶ Frege's central aim was to establish the truth of *logicism*: the truths of arithmetic are reducible to truths of logic, the falsehoods of arithmetic are reducible to falsehoods of logic.
- ▶ A subsidiary aim was to knock down competing philosophies of mathematics, especially Kant's.
- ▶ But he agreed with Kant on some points:

  > *I have no wish to incur the reproach of picking petty quarrels with a genius to whom we must all look up with grateful awe; I feel bound, therefore, to call attention also to the extent of my agreement with him, which far exceeds my disagreement. ... In calling the truths of geometry synthetic and a priori, he revealed their true nature. ... His point was that there are such things as synthetic judgements a priori, whether they are to be found in geometry only, or in arithmetic as well, is of less importance. (§89)*

Objections to Kant 1: infinite formulae

- For Frege, the *postulates* of arithmetic – the individual equations and inequations – are infinite in number.

  > *We must distinguish numerical formulae, such as $2 + 3 = 5$, which deal with particular numbers, from general laws, which hold good for all whole numbers. The former are held by some philosophers to be unprovable and immediately self-evident like axioms. Kant declares them to be unprovable and synthetic, but hesitates to call them axioms because they are general and because the number of them is infinite. Hankel justifiably calls this conception of infinitely numerous unprovable primitive truths incongruous and paradoxical. The fact is that it conflicts with one of the requirements of reason, which must be able to embrace all first principles in a survey. (§5)*

Objections to Kant 1: infinite formulae

- Human beings are finite, but there are infinitely many individual equations and inequations, and we cannot grasp an infinity of distinct facts.
- There cannot be infinitely many equations at the base of mathematics.
- There must be a finite number, which we are capable of surveying.
- It's not clear that this objection has much force: although there are infinitely many truths of e.g. addition, the rule by which we perform addition may be finitely graspable.

Objections to Kant 2: large numbers

- ► Frege thought that Kant's account was especially implausible for large numbers:

  > is it really self-evident that
  > $135664 + 37863 = 173527$?
  > It is not; and Kant actually urges this as an argument for holding these propositions to be synthetic. Yet it tells rather against their being unprovable; for how, if not by means of proof, are they to be seen to be true, seeing that they are not immediately self-evident?
  > ...
  > Kant, obviously, was thinking only of small numbers. So that for large numbers the formulae would be provable, though for small numbers they are immediately self-evident through intuition. (§5)

Objections to Kant 2: large numbers

- It does seem implausible that we have an immediate grasp of such large numbers.
- But Kant needn't allow that we have direct intuitions of such numbers.
- Rather, they are *grounded* in intuition. E.g. we can grasp smaller numbers and operations in order to grasp larger numbers.

Objections to Kant 3: generalisation

- Even if we can grasp particular arithmetic truths in intuition, how do we progress from these to generalisations?

  > In geometry, therefore, it is quite intelligible that general propositions should be derived from intuition; the points or lines or planes which we intuit are not really particular at all, which is what enables them to stand as representatives of the whole of their kind. But with the numbers it is different; each number has its own peculiarities. To what extent a given particular number can represent all the others, and at what point its own special character comes into play, cannot be laid down generally in advance. (§17)

Objections to Kant 3: generalisation

- Consider the following sentences:

  $1 + 3 = 3 + 1$
  $4 + 3 = 3 + 4$
  $2 + 5 = 5 + 2$

- Perhaps we can grasp each of these in isolation.

- But how do we jump from here to a grasp of the commutative law for addition, of which these are all instances?

Axiomatisation

- These 3 objections all target the individual justifications that Kant offers for our grasp of arithmetic, whether that is a grasp of infinitely many sentences, large numbers, or general laws.
- It is tempting to defend Kant on the basis of his views about axiomatisation. As we saw last week, he held that there is no axiomatisation of arithmetic.
- In his 1789 commentary on the *Critique*, Johann Schultz began to axiomatise arithmetic, but his attempt was incomplete, e.g. you couldn't prove any generalities, and he ignored multiplication.
- In his 1861 textbook, Hermann Grassmann built on Schultz's effort, but still ignored inequalities.

Axiomatisation

- Even if these early attempts at axiomatisation failed, we now know that it is possible.
- Does this meet Frege's objections?
- Not really: it just pushes the problem back to our grasp of the axioms.
- Frege was well aware of this, and wanted to attack Kant's position more fundamentally.
- He wanted to attack the claim that arithmetic depends on the spatio-temporal structure of experience.

Objections to Kant 4: intuitions

- First, in a related objection, Frege objected that we cannot *immediately* grasp a huge number in intuition like 10^{80}: the number of baryons in the observable universe.

- It does seem that such *immediate* recognition isn't possible, but it doesn't follow that a *mediate* grasp isn't possible.

- Further, as an *ad hominem*, Frege agreed with Kant that geometry relies on intuitions about space.

- But, for every number N we can entertain, however large, there will be a regular N-gon.

- And we know that there will be geometric theorems true of this shape, e.g. that the sum of its interior angles will be $(N - 2)\pi$.

Objections to Kant 5: law of contradiction

- ▶ Frege thought that consideration of the law of contradiction favoured his position.

  > *For purposes of conceptual thought we can always assume the contrary of some or other of the geometrical axioms, without involving ourselves in any self-contradictions ... Can the same be said of the fundamental propositions of the science of number? Here, we have only to try denying any one of them, and complete confusion ensues. Even to think at all is no longer possible. (§14)*

Objections to Kant 5: law of contradiction

- The thought is that we can imagine worlds with radically different geometries (we now know of hyperbolic geometries), but can we really imagine a world in which 2+2=5? Doesn't it pull against the thought (which Kant accepts) that mathematics is necessary?

- Reply: appeal to non-standard arithmetics; retreat to another notion of analyticity e.g. in terms of dependence on objects or non-ampliativity.

The nature of arithmetic

- ► For these reasons, Frege thought that arithmetic was much more general than geometry.

  > *The wildest visions of delirium, the boldest inventions of legend and poetry, where animals speak and stars stand still, where men are turned to stone and trees turned into men, where the drowning haul themselves up out of swamps by their topknots – all these remain, so long as they remain intuitable, still subject to the axioms of geometry. ... The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. (§14)*

The nature of logic

- ▶ Frege is beginning to suggest that arithmetic is so general that it is in fact a *part* of logic: the most general science.
- ▶ If we could show that arithmetic is a part of logic, we could explain its applicability to 'everything thinkable'.

> *The truths of arithmetic would then be related to those of logic in much the same way as the theorems of geometry to the axioms. Each one would contain concentrated within it a whole series of deductions for future use and the use of it would be that we need no longer make the deductions one by one, but can express simultaneously the result of the whole series. ... [This] will suffice to put an end to the widespread contempt for analytic judgements and to the legend of the sterility of logic. (§17)*

The nature of logic

- That we could ever hope for logic to play this role in the foundations of arithmetic is only thinkable given Frege's innovations in the *Begriffschrift*.
- In traditional Aristotelian syllogistic, there is a sense in which we are merely 'taking out of the box again what we have just put into it' (§88).
- Nevertheless, Aristotle introduced the crucial idea of *variables* to express generality.
- We can express *syllogisms* such as *Barbara*: All Fs are Gs, All Hs are Fs; *so* All Hs are Gs. Or *Darii*: All Fs are Gs, Some Hs are FS; *so* Some Hs are Gs.
- This was still very much Kant's notion of logic. It resembles *conceptual containment*.

Dependence on objects

- ▶ Frege's logic allows for conclusion that 'extend our knowledge' (§88).
- ▶ One hallmark of ampliative, synthetic reasoning, we have seen, is dependence on objects.
- ▶ It is the use of *variables* that allows this: we can, to use Dummett's example, extract the distinct concepts of murder and suicide from the sentence 'Cassius killed Caesar': 'x killed y', 'x killed x'.
- ▶ Aristotle's (and Kant's) logic was essentially the *monadic fragment* of Frege's.
- ▶ The power of Frege's insight is captured in the metatheory: polyadic quantified logic is not decidable.

Objections to Kant 6: analytic/ synthetic distinction

- ▶ We might think that the use of variables here reveals a certain dependence on *objects*.

- ▶ This all looks quite Kantian: arithmetic is logic, logic extends knowledge, so arithmetic extends knowledge. Anything that extends knowledge is ampliative, so arithmetic is ampliative and so synthetic.

- ▶ But Frege demands that arithmetic is still analytic, because independent of intuition.

- ▶ He wants to show

    > *how pure thought (regardless of any content given through the senses or even given a priori through an intuition) is able, all by itself, to produce from the content which arises from its own nature judgements which at first glance seem to be possible only on the grounds of some intuition. (Begriffschrift, §23)*

Objects

- Frege admitted that he held a different notion of *object* to Kant.
- For Kant, we only grasp objects through intuition.
- Frege needed to eliminate this appeal to intuition, and to show that logic alone can introduce us to objects.
- This is the major task of his logicism.

Three principles

- Frege begins the *Grundlagen* by listing three principles which will inform the book:

  > *In the enquiry that follows, I have kept to three fundamental principles:*
  >
  > > *always to separate the psychological from the logical, the subjective from the objective;*
  > > *never to ask for the meaning of a word in isolation, but only in the context of a proposition;*
  > > *never to lose sight of the distinction between concept and object.*

Principle 3: Concept and object

- For Frege, an *object* is an entity referred to by a *singular term*.
- A *concept* is an entity referred to by a *predicate*.
- Frege draws a sharp distinction between object and concept: nothing can be both an object and a concept, and it is nonsense to say of an object that it is a concept, or *vice versa*.
- The distinction between sense and reference was not clear to Frege at the time of writing *Grundlagen*.

Concepts

- Consider the sentence 'Frege is a logician'. If we delete the name 'Frege' from this sentence and replace it with a variable, we get the first-level predicate 'x is a logician'. A first-level predicate picks out a first-level concept.

- Now consider the sentence '$\forall x$ x is a logician'. If we delete the first-level predicate 'x is a logician and replace it with a variable, we get the second-level predicate '$\forall x F x$'. A second-level predicate picks out a second-level concept.

- Concepts, of any order, have *extensions*. These are the things falling under them. The things falling under a first-level concept are objects. the things falling under a second-level concept are first-order concepts.

Objects

- It may *appear*, when we work in monadic first-order logic, that we are reasoning about concepts, e.g. All *A*s are *B*s.
- But this is because we have *suppressed* the variable. And, when we move to polyadic logic, this suppression is impossible.
- And the notion of concept presupposes that of object:

  > *Instead of putting a judgement together out of an individual as subject and an already previously formed concept as predicate, we do the opposite and arrive at a concept by splitting up the content of possible judgement. (Posthumous Writings, p. 17)*

Objects

- For Frege, an object is anything that can be united with a subject to form a judgement.
- An object is anything picked out by a singular term. And what it is to be a singular term is to function linguistically in the correct manner.
- There is no requirement that objects are subject to a mind-imposed structure.
- Kant's criterion for objecthood was *psychological* and Frege's is *semantic*.
- This notion of objecthood is captured by the Context principle – one of the three principles of the *Grundlagen*.

Principle 2: Context principle

- The *context principle* instructs us 'never to ask for the meaning of a word in isolation but only in the context of a proposition'.

- Michael Dummett argues that the context principle marks a fundamental shift in philosophy:
  the *linguistic turn* (compare: Kant's Copernican turn).

  > *The thesis that it is only in the context of a sentence that a word has meaning: the investigation therefore takes the form of asking how we can fix the senses of sentences containing words for numbers. (Dummett 1993: 5)*

Principle 1: Anti-psychologism

- One use of the Context Principle is to rule out psychologism:

    only be adhering to this can we avoid a physical view of number without slipping into a psychological view of it (§106)

- We might think that numbers exist but cannot be physical, so they must be ideas.

- Frege attricutes this idea to Schloemilch, and goes on to destroy it.

- First, ideas are private, whereas numbers don't seem to be. If your concept of the number 1 differs from mine, then it's a kind of miracle that we manage to communicate.

- Second, some numbers have never been thought of it, perhaps because they are too large. But we don't want to deny that these numbers exist.

- *Grundlagen* is widely regarded to have demolished psychologism.

Conclusion

- We've seen Frege's criticisms of Kant, and started to sketch his own.
- Like Kant, Frege thought that geometry was synthetic *a priori*, but that arithmetic was analytic *a priori*.
- His notion of the analytic was independence of intuition.
- Arithmetic is true independently of intuition because it is reducible to logic.
- It remains to be seen *how* mathematics is reducible to logic, especially how logic can *introduce* objects.
- That's our topic for next week.

Implicit definition

- An *explicit* definition is one that defines an expression in terms of previously understood expressions, e.g. a vixen is a female fox.

- An *implicit* definition is a functional definition that defines an expression in terms of its role, e.g. Jack the Ripper is whoever committed the Whitechapel murders.

- It may be that nobody committed the Whitechapel murders, or that many people did. But, if someone committed the Whitechapel murders, then that person is being defined as 'Jack the Ripper'.

- One role of the context principle is to legitimise implicit definitions.

Implicit definition in *Grundlagen*

- Frege considers a number of implicit definitions of number. His most famous suggestion is now known as *Hume's Principle*:

    HP $NxFx = NxGx \leftrightarrow F \sim G$

- Here, N is a term-forming operator and 'NxFx' denotes the number of Fs.

- $F \sim G$ abbreviates:

    $$\exists R(\forall x(Fx \rightarrow \exists! y(Gy \wedge Rxy) \wedge$$
    $$\forall y(Gy \rightarrow \exists! x(Fx \wedge Rxy))))$$

- In words: there is a relation which relates each F to exactly one G and each G to exactly one F.

- When such a relation exists, the Fs and the Gs are equinumerous, so the number of Fs is identical to the number of Gs.

The significance of HP

- Remarkably, if we consider a theory which is built on second-order logic and has HP as its only axiom, we can derive all of the second-order Peano axioms as theorems.
- Call this theory *Frege Arithmetic* and the result *Frege's Theorem*.
- Frege does not prove Frege's Theorem in *Grundlagen*.
- The proof was conjectured by Crispin Wright and proved by, amongst others, George Boolos (see his 'On the proof of Frege's Theorem', 1988).

Frege's Theorem

- To prove Frege's Theorem, we need to show that all of the axioms of PA^2 are theorems of Frege Arithmetic.
- One axiom of PA^2 states that 0 is not the successor of any number.
- We could express this as:

    Ax $\neg\exists x(Nx \wedge Px0$

    where 'Nx' expresses 'x is a number' and 'Pxy' expresses 'x is the predecessor of y'.
- We can define *predecession* in the following way:

    $$Pmn =_{df} \exists F\exists x(Fx \wedge n = NzFz \wedge m = Nz(Fz \wedge z \neq x))$$

    In words: m is the predecessor of n just if n is the number of Fs, for some F, and m is the number of Fs excluding one object.