PART II PAPER 07:
MATHEMATICAL LOGIC

SYLLABUS

- First-order and second-order logic: completeness, compactness, conservativeness, expressive power and Löwenheim-Skolem theorems.
- First and second order theories: categoricity, non-standard models of arithmetic.
- Set theory: embedding mathematics in set theory, the cumulative iterative hierarchy, elements of cardinal and ordinal arithmetic, the axiom of choice.
- Recursive functions and computability: decidability, axiomatizability, Church’s thesis, Gödel’s incompleteness theorems, Hilbert’s programme.

COURSE OUTLINE

This course aims to put the student in a position to assess the philosophical significance of some major results in mathematical logic. Alert attendance at lectures should be considered essential. But the emphasis is not only on the rigorous proof of results but also on philosophical reflection about them.

The first part of the course tackles the ideas of a formal logic and a formal theory with particular attention to the similarities and differences between first-order and second-order logic and arithmetic.

The next part studies set theory, its axioms and motivating conceptions, and the idea that all of mathematics can be embedded in set theory.

In the third part of the course on recursive functions the informal notion of a computable function is described and the relationship, embodied in Church’s thesis, between this informal notion and the precise notion of a recursive function is examined. All this leads up to the understanding of Gödel’s incompleteness theorems for arithmetic.

Prerequisites

Study of mathematics beyond GCSE or its equivalent is highly desirable, not for the content but for the familiarity with mathematical modes of thought. It is essential that candidates who have not taken Part IA Logic should familiarise themselves with an introductory logic text (see Part IA reading list). The emphasis is on just a few big results and their philosophical significance, not on the nitty gritty of grinding out lots of proofs. And although the examination includes an opportunities to show off an ability to outline proofs of those big results, you can take the paper by concentrating on philosophical issues. This paper combines well with Philosophical Logic because both include topics in the philosophy of mathematics.

Objectives

Students will be expected to:
1. Study issues in mathematical logic at an advanced level.
2. Acquire a sophisticated understanding of the scope, purpose and natures of logic.
3. Refine their power of philosophical analysis and argument through study of these ideas.

Preliminary Reading

On basic logic:


On set theory:


On arithmetic, computability etc.


For classic essays on some of the conceptual issues discussed in the course see:

READING LIST

Background

General Formal Surveys

Two very useful, more discursive surveys, standing back a bit from the nitty gritty of proofs, but trying to give a sense of how results fit together with an indication of their wider significance, are:

ROGERS, Robert, Mathematical Logic and Formalized Theories (Amsterdam: North-Holland, 1971).
WOLF, Robert S., A Tour through Mathematical Logic (Washington: Mathematical Association of America, 2005).

Rogers’s now rather old book is very useful and very accessible though relatively introductory. Wolf’s newer book goes further but is a rather bumpier ride because it’s somewhat uneven in level of difficulty (though he gives some useful proof sketches). These books will make very useful companions to formal work over the year, and could be especially helpful whenever you feel in danger of not seeing the wood for the trees.

The Way the Rest of This Reading List is Structured

For formal expositions, we’ll list some possible alternatives, because different presentations are to the taste of different readers.

For philosophical topics, we usually distinguish core readings – in something like a sensible reading order – from a selection of possible further readings. Such divisions are inevitably somewhat arbitrary, and different supervisors will want to take different views about what is basic – needed to make a shot at a supervision essay – and what pushes on the debate rather further.

FIRST ORDER LOGIC

Formal Expositions

The key things you’ll need to understand are the ideas of soundness/completeness theorems, the compactness theorem, and the Löwenheim-Skolem theorems – and how to prove them.

The books by Chiswell/Hodges and by Leary already mentioned of course cover first-order logic in an accessible way. And almost any standard middle or advanced level text will cover the needed ground. But a stand-out presentation is:


For a more discursive introduction to some main ideas see:

ROGERS, Robert, Mathematical Logic and Formalized Theories (Amsterdam: North-Holland, 1971), chs. 2 & 3.

For another overview treatment highlighting the main ideas, though in more detail, you could see:

BOOLOS, George, John BURGESS, and Richard C. JEFFREY, Computability and Logic. 4th or 5th ed. (Cambridge: Cambridge University Press, 2002; 2007), chs. 9, 10 & 12-14. The 5th ed. is also available online at: http://lib.myilibrary.com?ID=238992

And finally let’s mention two more standard full-dress textbook treatments for enthusiasts:

ENDERTON, Herbert B., Mathematical Introduction to Logic. 2nd ed. (San Diego, CA: Harcourt/Academic Press, 2002), ch. 2 'First-order logic'.

Philosophical Issues Arising: Skolem’s Paradox

For basic discussion see:


Also see, for slightly more brisk discussion, the opening third of:


For further reading, try the rest of Bays’s article, and some of:

SECONd ORDER LOGIC

Formal Expositions

You need some sense of the difference between first- and second-order logic in terms of axiomatizability, compactness, L-S theorems etc. You’ll need to understand why e.g. second-order Peano arithmetic is categorical and first-order Peano arithmetic isn’t.

For a useful introductory overview, see:

ROGERS, Robert, Mathematical Logic and Formalized Theories: A Survey of Basic Concepts and Results (Amsterdam: North-Holland, 1971), ch. 4, sects. 1-5.

Another introductory formal survey is:


But the classic modern presentation is no doubt:

SHAPIRO, Stewart, Foundations without Foundationalism (Oxford: Oxford University Press, 1991), chs. 3-5. Also available online at: http://doi.org/10.1093/0198250290.001.0001

(Shapiro’s book is subtitled “A Case for Second-Order Logic”). That would give you more than enough. If you prefer a much brisker overview, see:

BOOLOS, George, John BURGESS, and Richard Jeffrey, Computability and Logic. 4th or 5th ed. (Cambridge: Cambridge University Press, 2002; 2007), ch. 22 'Second-order logic'. The 5th ed. is also available online at: http://lib.myilibrary.com?ID=238992

Enthusiasts who are looking for other textbook treatments can try:

ROBBIN, Joel W., Mathematical Logic: A First Course (New York: Benjamin, 1969), ch. 6 'Second-order logic'.

VAN DALEN, Dirk, Logic and Structure. 4th ed. (Berlin: Springer, 2004), ch. 4 'Second order logic'.

Finally, for more discussion of first- vs. second-order arithmetic, you should look at:

SMITH, Peter, An Introduction to Gödel's Theorems (Cambridge: Cambridge University Press, 2007 - 1st ed.; 2013 - 2nd ed.), ch. 22 (ch. 29 in 2nd ed.) 'The Diagonalization Lemma', 2nd ed. also available online at: http://doi.org/10.1017/CBO9781139149105

And for a discussion of the sort of non-standard models that first-order arithmetic can have, see:

BOOLOS, George, John BURGESS, and Richard Jeffrey, Computability and Logic. 4th or 5th ed. (Cambridge: Cambridge University Press, 2002; 2007), ch. 25 'Nonstandard models'. The 5th ed. is also available online at: http://lib.myilibrary.com?ID=238992

Philosophical Issues Arising: 1, On the Status of Second-Order Logic as Logic

Is second-order logic just set-theory in disguise (with the second-order quantifiers running over sets)? That’s the view of:

QUINE, Willard V.O., Philosophy of Logic. 2nd ed. (Cambridge, MA: Harvard University Press, 1986), ch. 5 'The scope of logic'.

For discussion see:


For further reading, see:


**Philosophical Issues Arising: 2, The Connections with Plural Quantification and Natural Language**

George Boolos has argued that we can "tame" second-order logic (and see it as genuinely part of logic) by interpreting second-order quantifiers as (akin to) plural quantifiers. For a basic exchange, see:


For further discussion see:


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**SET THEORY**

**ZFC AND CLOSELY RELATED SET THEORIES**

**Formal Expositions**

For an informal outline of some key ideas, see:


WOLF, Robert S., *A Tour through Mathematical Logic* (Washington, DC: Mathematical Association of America, 2005), ch. 2 'Axiomatic set theory'.

The next is a classic and pleasingly slim volume that also introduces some key ideas in a very accessible way:


The opening chapters of the following classic are still very worth reading:


But three texts stand out:


Potter's version of set theory - coming to be known as "the Scott-Potter theory" - is elegant but somewhat non-standard; Devlin is a beautifully written presentation of the standard theory. Here's another modern text that is written in a relaxed style (there are even jokes), and is often extremely helpful in the way it introduces concepts and theorems:


Finally, it is worth following up Fraenkel and Bar-Hillel's semi-historical review by looking at:

and perhaps also:

Those who find the historical stories fascinating - and they illuminate why one particular set theory has ended up as the canonical one - can follow up Ferreirós by dipping into at least the first half of the longer story as told by:

Philosophical Issues Arising: 1, Set Theory as a Foundation for Mathematics

In what sense can we say that set theory "provides a foundation for" mathematics?

GIQUINTO, Marcus, The Search for Certainty (Oxford: Oxford University Press, 2002), part I; and part V, sects. 1 & 2. Also available online at: http://lib.myilibrary.com/?id=91428

Gives an introductory discussion of what set theory is supposed to do for us. The set theory texts above have things to say as they go along. For further discussion see:
MADDY, Penelope, Naturalism in Mathematics (Oxford: Oxford University Press, 1997), ch. 2 'Realism'. Also available online at: http://doi.org/10.1093/0198250754.001.0001


There are some subversive remarks too in:

See also:


GÖDEL’S FIRST INCOMPLETENESS THEOREM

Formal Expositions

For a nice introduction (in a splendidly sane short book, which you should eventually read all of), see:
And for another introductory survey see:


There's a bit more detail again in:


But for a full-dress proof with all the trimmings see:


Those recommendations on the formalities should more than suffice for philosophical purposes. But some might like the rather different approach of the enviably elegant:


which will particularly appeal to the mathematically minded.

**Note:** By the way that Gödel proved his First Theorem in 1931, before the beginnings of the general theory of computability really got underway in 1936: the original version of the Theorem appeals only to the restricted notion of a "primitive recursive" function. Many modern books, however, approach things in a non-historical order, first explaining the general theory of computability, and then moving on to Gödel's Theorem. Two notable books which do things this way around are:


Philosophical Issues Arising: 1, Minds and Machines


Lucas famously argues that Gödel's theorem shows that minds are not machines. (It is not really essential, but might help if you know what Turing machine is before you start reading this debate). For a classic riposte, see:

PUTNAM, Hilary, *Mind, Language and Reality, Philosophical Papers*. Vol. 2 (Cambridge: Cambridge University Press, 1975), ch. 18 'Minds and Machines'. Also available online at: [http://doi.org/10.1017/CBO9780511625251](http://doi.org/10.1017/CBO9780511625251)

Others have tried to rescue Lucas's argument, in particular:


For a stern critique of that see:


(There's much more on Penrose to be found in the same issue of *Psyche* at: [www.theassc.org/vol_2_1995_1996](http://www.theassc.org/vol_2_1995_1996))

For other related discussion see:

GÖDEL, Kurt, *Collected Works*. Vol. III (Oxford: Oxford University Press, 1995), pp. 304-23 'Some basic theorems in the foundations of mathematics and their philosophical implications'. Also available on Moodle. [This, the "Gibbs Lecture" from 1951 is not easy but is remarkably rich]


SMITH, Peter, *An Introduction to Gödel's Theorems* (Cambridge: Cambridge University Press, 2007 - 1st ed.; 2013 - 2nd ed.). 2nd ed. also available online at: [http://doi.org/10.1017/CBO9780511625237](http://doi.org/10.1017/CBO9780511625237). [Sect. 28.6 (37.6 in 2nd ed.) - relates to the argument in Gödel's paper which springs from the Second Incompleteness Theorem]

Philosophical Issues Arising: 2, Is the Notion of Natural Number Open-Ended?


GÖDEL'S SECOND INCOMPLETENESS THEOREM

Formal Expositions

GEORGE, Alexander, and Daniel J. VELLEMAN, Philosophies of Mathematics (Oxford: Blackwell, 2001), ch. 7 'The Incompleteness Theorems'.

could be a useful place to start. And for quite a bit more detail see:


That should put you in a position to appreciate Boolos's wonderful jeu d'esprit:


For useful commentary, see:


HILBERT'S PROGRAMME

As we'll see, the main philosophical issue arising from Gödel's Second Theorem (at least as far as this paper is concerned) is its impact on Hilbert's Programme. For a very good introduction to Hilbert, see:

GIAQUINTO, Marcus, The Search for Certainty (Oxford: Oxford University Press, 2002), part IV, sects. 3 & 4. Also available online at: http://lib.myilibrary.com/?id=91428

But do read the man himself:


For further elaboration see also:

GEORGE, Alexander, and Daniel J. VELLEMAN, Philosophies of Mathematics (Oxford: Blackwell, 2001), ch. 6 'Finitism'.

And for further discussion, see:


Philosophical Issue Arising: What was Hilbert's Programme? Do Gödel's Incompleteness Theorems Undermine It?

A standard answer to the second question is given by:

GIAQUINTO, Marcus, The Search for Certainty (Oxford: Oxford University Press, 2002), ch. 5, sect. 2 'Underivability of 'Consistency''. Also available online at: http://lib.myilibrary.com/?id=91428

And similarly by:


For dissent see:


For more discussions see:


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**RECURSIVE FUNCTIONS AND COMPUTABILITY**

**Expositions**

What is Turing computable function? What is recursive function? Why are they the same class of functions? For explanations, see:


For other alternatives, see:

BOOLOS, George, John BURGESS, and Richard C. JEFFREY, *Computability and Logic*. 4th or 5th ed. (Cambridge: Cambridge University Press, 2002; 2007), chs. 3-8. The 5th ed. is also available online at: [http://www.myilibrary.com?ID=238992](http://www.myilibrary.com?ID=238992). Though many think the treatment of the same chapters of the 3rd ed. - when the authors were just Boolos and Jeffrey - is nicer.


ROGERS, Hartley, *Theory of Recursive Functions and Effective Computability* (Cambridge, MA: MIT Press, 1987). [Another old classic from 1967 which is well worth reading the first chapter of, especially sects. 1.1-1.7]

**Philosophical Issue Arising: What is the Status of Church's Thesis?**

It is a mathematical theorem that a function is Turing computable if and only if it is recursive (if and only if it register computable, if and only if it is Herbrand-Gödel computable etc.). Different attempts to regiment the intuitive notion of a computable function all converge. Church's Thesis (a.k.a. the Church-Turing Thesis) claims that indeed the *intuitively* computable functions are just the Turing computable/recursive functions.

For some initial clarifications, see:


Then read:


SMITH, Peter, *An Introduction to Gödel's Theorems* (Cambridge: Cambridge University Press, 2007 - 1st ed.; 2013 - 2nd ed.). 2nd ed. also available online at: [http://doi.org/10.1017/CBO9781139149105](http://doi.org/10.1017/CBO9781139149105). [Ch. 35 (ch. 45 in 2nd ed.) 'Halting problems', takes an opinediated minority line]

See also:


There's an interesting local sub-debate here:


We welcome your suggestions for further readings that will improve and diversify our reading lists, to reflect the best recent research, and important work by members of under-represented groups. Please email your suggestions to phillib@hermes.cam.ac.uk including the relevant part and paper number. For information on how we handle your personal data when you submit a suggestion please see https://www.information-compliance.admin.cam.ac.uk/data-protection/general-data.