(a) A truth-function is a function that takes one or many truth-values as inputs and gives a truth-value as an output.

(ii) A set of sentences $P_1, P_2, \ldots, P_n$ tautologically entails a sentence $Q$ if and only if there is no assignment of truth values to the atoms of $P_1, P_2, \ldots, P_n$ and $Q$ on which $P_1, P_2, \ldots, P_n$ are all true and $Q$ is false.

(iii) The material conditional $\supset$ is the two-place propositional connective such that $P \supset Q$ is true unless $P$ is true and $Q$ is false.

(iv) The metalanguage is the language in which one describes properties and relations of elements of the object language e.g. tautological entailment.

(b) A system that is sound but not complete is the system that allows nothing to be deduced from anything.

(ii) A system that is complete but not sound is the system that allows anything to be deduced from anything.

(c) To say that a truth-function can be expressed by $\land$, $\lor$ and $\neg$ is to say that every truth function is some combination of these truth-functions. To prove it: consider a truth-function $G$ that takes $n$ truth-values to a truth-value. Then $G$'s truth-table is as follows:

<table>
<thead>
<tr>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$\ldots$</th>
<th>$P_n$</th>
<th>$G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>$\ldots$</td>
<td>T</td>
<td>$X_1$</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>$\ldots$</td>
<td>T</td>
<td>$X_2$</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>$\ldots$</td>
<td>T</td>
<td>$X_3$</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>$\ldots$</td>
<td>F</td>
<td>$X_4$</td>
</tr>
<tr>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>$\ldots$</td>
<td>F</td>
<td>$X_k$</td>
</tr>
</tbody>
</table>

Each of $X_1, X_2, \ldots, X_k$ is either T or F. Consider all the rows $j_1, j_2, \ldots, j_q$ in which $X_j = T$. Call each such row a good row. Now define for each $i$:
\[ H(P_i, m) = P_i \text{ if } P_i \text{ is } T \text{ in row } m \]
\[ H(P_i, m) = \neg P_i \text{ if } P_i \text{ is } F \text{ in row } m \]
\[ E(m) = H(P_1, m) \land H(P_2, m) \land \ldots \land H(P_n, m) \]

Then \( G(P_1, \ldots, P_n) = E(j_1) \lor E(j_2) \lor \ldots \lor E(j_3) \) where \( j_1, j_2, \ldots, j_3 \) index all and only the good rows. And clearly \( E(j_1) \lor E(j_2) \lor \ldots \lor E(j_3) \) involves only \( \land, \lor \) and \( \neg \).

2. (a)

(i) \[ Lkj = \neg M \]

(ii) \[ \forall x (\neg \exists y Lxy \supset (\neg Lxj \land \neg Lxk)) \]

(iii) \[ \neg Mj \supset \forall x (Lkx = x = j) \]

(iv) \[ \exists x \exists y \forall z (((z \neq y \lor z \neq x) \land Lzj) = (z = x \lor z = y)) \land Mx = \neg My \]

(v) \[ \exists x \exists y \forall z ((Bz = z = x) \land (Mz = z = y) \land (Lzy \supset \neg Lzx)) \]

(vi) \[ \forall x \forall y ((Bx \land By \land y \neq x \land Lxk \land Lyk) \supset (Lxy \land Lyx)) \]
(b) (i)

\[ M_j \land \forall x (L_{jx} \equiv L_{kx}) \]

\[ \forall x (M_{xj} \circ L_{kx}) \]

\[ \neg L_{ij} \]

\[ M_j \]

\[ \forall x (L_{jx} \equiv L_{kx}) \]

\[ L_{ij} \equiv L_{kj} \]

\[ M_j \circ L_{kj} \]

\[ \neg M_j \]

\[ L_{kj} \]

\[ \neg \neg \]

\[ L_{ij} \]

\[ L_{kj} \]

\[ L_{ij} \]

\[ \neg \neg \]
(b) (ii)

\[ \forall x (M(x) \equiv x = k) \land B \]

\[ \neg \forall x ((M(x) \land B(x)) \equiv x = k) \]

\[ \exists x \forall x ((M(x) \land B(x)) \equiv x = k) \]

\[ \forall x (((M(x) \land B(x)) \equiv x = k) \land B) \]

\[ \neg M(a) \equiv a = k \]
(b) (iii)

\[ \forall x \exists y (L_y \land B_y \land L_y \land B_y) \Rightarrow TL_{KB} \]

\[ \forall y (L_y \land B_y \land L_y \land B_y) \Rightarrow TL_{KB} \]

\[ L_{KB} \Rightarrow T_{KB} = j \]

\[ TL_{KB} \]

\[ TL_{KB} \]

\[ L_{KB} \Rightarrow B = j \]

\[ \forall x (\exists y (L_y \land B_y \land L_y \land B_y) \Rightarrow TL_{KB}) \]

\[ \exists y (L_y \land B_y \land L_y \land B_y) \Rightarrow TL_{KB} \]

\[ TL_{KB} \]

\[ TL_{KB} \]

\[ TL_{KB} \]

\[ TL_{KB} \]

\[ TL_{KB} \]

\[ TL_{KB} \]
3.

<table>
<thead>
<tr>
<th></th>
<th>R</th>
<th>S</th>
<th>T</th>
<th>NT</th>
</tr>
</thead>
<tbody>
<tr>
<td>x is y's father</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>x is a brother of y</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>x is y's only sibling</td>
<td>N</td>
<td>Y</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>x and y have no common ancestor</td>
<td>N</td>
<td>Y</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>x is an ancestor of y</td>
<td>N</td>
<td>N</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>x loves y = y loves x</td>
<td>Y</td>
<td>Y</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>x loves y v y loves x</td>
<td>N</td>
<td>Y</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>∀z (x loves z = z loves y)</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>x loves John = y loves Jane</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>Majority pref.</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
</tbody>
</table>

4.

(a) This only holds if 5 black socks and 1 white sock so the answer is 0

(b) \( \Pr (2B \mid 1B) = \frac{\Pr (2B)}{\Pr (1B)} = \frac{0.25}{0.75} = \frac{1}{3} \)

(c) \( \Pr (2B \mid 1BM) = \frac{\Pr (2B \land 1BM)}{\Pr (1BM)} = \frac{13/196}{27/196} = \frac{13}{27} \)

(d) Use the formula:
\[
\Pr (A \mid T) = \frac{\Pr (T \mid A) \Pr (A)}{[\Pr (T \mid A) \Pr (A) + \Pr (T \mid B) \Pr (B)]} = \frac{(1/2 \times 1/2)}{[(1/2 \times 1/2) + (2/3 \times 1/2)]} = \frac{3}{7}
\]

(e) use the same formula as in (d): the answer is
\[
\frac{(0.9 \times 0.2)}{(0.9 \times 0.2) + (0.1 \times 0.8)} = \frac{9}{13}
\]