1A Logic — 2016 Model Answers

Section A

1. (a) Show each of the following using the proof system for TFL described in *forallx*.

(i) $\neg(P \leftrightarrow Q) \vdash \neg(Q \leftrightarrow P)$

1	$\neg(P \leftrightarrow Q)$	
2	$\quad Q \leftrightarrow P$	
3	$\quad\quad P$	
4	$\quad\quad Q$	$\leftrightarrow$E, 2, 3
5	$\quad\quad Q$	
6	$\quad\quad P$	$\leftrightarrow$E, 2, 5
7	$\quad P \leftrightarrow Q$	$\leftrightarrow$I, 3–4, 5–6
8	$\quad \bot$	$\bot$I, 7, 1
9	$\neg(Q \leftrightarrow P)$	$\neg$I, 2–7

(ii) $P \wedge \neg Q) \vee R \vdash Q \rightarrow (P \vee \neg\neg R)$

1	$P \wedge \neg Q) \vee R$	
2	$\quad Q$	
3	$\quad\quad P \wedge \neg Q$	
4	$\quad\quad P$	$\wedge$E, 3
5	$\quad\quad P \vee \neg\neg R$	$\vee$I, 4
6	$\quad\quad R$	
7	$\quad\quad\quad \neg R$	
8	$\quad\quad\quad \bot$	$\bot$I, 6, 7
9	$\quad\quad \neg\neg R$	$\neg$I, 7–8
10	$\quad\quad P \vee \neg\neg R$	$\vee$I, 9
11	$\quad P \vee \neg\neg R$	$\vee$E, 1, 3–5, 6–10
12	$Q \rightarrow (P \vee \neg\neg R)$	$\rightarrow$I, 2–11

(iii) $\neg((P \rightarrow \neg Q) \vee \neg R) \vdash Q$

1	$\neg((P \rightarrow \neg Q) \vee \neg R)$	
2	$\neg(P \rightarrow \neg Q) \wedge \neg\neg R$	DeM, 1
3	$\neg(P \rightarrow \neg Q)$	$\wedge$E, 2
4	$\quad \neg Q$	
5	$\quad\quad P$	
6	$\quad\quad \neg Q$	R, 4
7	$\quad P \rightarrow \neg Q$	$\rightarrow$I, 5–6
8	$\quad \bot$	$\bot$I, 7, 3
9	$\neg\neg Q$	$\neg$I, 4–8
10	Q	DNE, 9

(iv) $\neg(P \to (Q \vee R)) \vdash P \wedge \neg R$

1	$\neg(P \to (Q \vee R))$	
2	$\quad \mid R$	
3	$\quad \mid \quad \mid P$	
4	$\quad \mid \quad \mid Q \vee R$	$\vee$I, 2
5	$\quad \mid P \to (Q \vee R)$	$\to$I, 3–4
6	$\quad \mid \bot$	$\bot$I, 5, 1
7	$\neg R$	$\neg$I, 2–6
8	$\quad \mid \neg P$	
9	$\quad \mid \quad \mid P$	
10	$\quad \mid \quad \mid \bot$	$\bot$I, 9, 8
11	$\quad \mid \quad \mid Q \vee R$	$\vee$I, 10
12	$\quad \mid P \to (Q \vee R)$	$\to$I, 9–11
13	$\quad \mid \bot$	$\bot$I, 12, 1
14	$\neg\neg P$	$\neg$I, 8–13
15	P	DNE, 14
16	$P \wedge \neg R$	$\wedge$I, 15, 7

(v) $\vdash P \leftrightarrow ((P \wedge \neg Q) \vee (P \wedge Q))$

1	$\quad P$	
2	$\quad\quad Q$	
3	$\quad\quad P \wedge Q$	$\wedge$I, 1, 2
4	$\quad\quad (P \wedge \neg Q) \vee (P \wedge Q)$	$\vee$I, 3
5	$\quad\quad \neg Q$	
6	$\quad\quad P \wedge \neg Q$	$\wedge$I, 1, 5
7	$\quad\quad (P \wedge \neg Q) \vee (P \wedge Q)$	$\vee$I, 6
8	$\quad (P \wedge \neg Q) \vee (P \wedge Q)$	TND, 2–4, 5–7
9	$\quad (P \wedge \neg Q) \vee (P \wedge Q)$	
10	$\quad\quad P \wedge \neg Q$	
11	$\quad\quad P$	$\wedge$E, 10
12	$\quad\quad P \wedge Q$	
13	$\quad\quad P$	$\wedge$E, 12
14	$\quad P$	$\vee$E, 9, 10–11, 12–13
15	$P \leftrightarrow ((P \wedge \neg Q) \vee (P \wedge Q))$	$\leftrightarrow$I, 1–8, 9–14

(b) Show each of the following using the proof system for FOL described in *forallx*. [50]

(i) $\forall x \forall y x = y \vdash \neg \exists x \exists y \neg x = y$

1	$\forall x \forall y x = y$	
2	$\quad \exists x \exists y \neg x = y$	
3	$\quad\quad \exists y \neg a = y$	
4	$\quad\quad\quad \neg a = b$	
5	$\quad\quad\quad \forall y a = y$	$\forall$E, 1
6	$\quad\quad\quad a = b$	$\forall$E, 5
7	$\quad\quad\quad \bot$	$\bot$I, 6, 4
8	$\quad\quad \bot$	$\exists$E, 3, 4–7
9	$\quad \bot$	$\exists$E, 2, 3–8
10	$\neg \exists x \exists y \neg x = y$	$\neg$I, 2–9

(ii) $\forall x \neg Rxx, Rab \vdash \exists x \exists y \neg x = y$

1	$\forall x \neg Rxx$	
2	Rab	
3	$\quad a = b$	
4	$\quad Raa$	=E, 3, 2
5	$\quad \neg Raa$	$\forall$E, 1
6	$\quad \bot$	$\bot$I, 4, 5
7	$\neg a = b$	$\neg$I, 3–6
8	$\exists y \neg a = y$	$\exists$I, 7
9	$\exists x \exists y \neg x = y$	$\exists$I, 8

(iii) $\forall x \exists y (Ryx \lor Qyx) \vdash \forall x (\exists y Ryx \lor \exists y Qyx)$

1	$\forall x \exists y (Ryx \lor Qyx)$	
2	$\exists y (Rya \lor Qya)$	$\forall$E, 1
3	$\quad Rba \lor Qba$	
4	$\quad\quad Rba$	
5	$\quad\quad \exists y Rya$	$\exists$I, 4
6	$\quad\quad \exists y Rya \lor \exists y Qya$	$\lor$I, 5
7	$\quad\quad Qba$	
8	$\quad\quad \exists y Qya$	$\lor$I, 7
9	$\quad\quad \exists y Rya \lor \exists y Qya$	$\lor$I, 8
10	$\quad \exists y Rya \lor \exists y Qya$	$\lor$E, 3, 4–6, 7–9
11	$\exists y Rya \lor \exists y Qya$	$\exists$E, 2, 3–10
12	$\forall x (\exists y Ryx \lor \exists y Qyx)$	$\forall$I, 11

(iv) $\forall x\exists yRxy, \exists x\exists yRxy \vdash \exists x\exists y\exists z(Rxy \wedge Ryz)$

1	$\forall x\exists yRxy$	
2	$\exists x\exists yRxy$	
3	$\quad \exists yRay$	
4	$\quad\quad Rab$	
5	$\quad\quad \exists yRby$	$\forall$E, 1
6	$\quad\quad\quad Rbc$	
7	$\quad\quad\quad Rab \wedge Rbc$	$\wedge$I, 4, 6
8	$\quad\quad\quad \exists z(Rab \wedge Rbz)$	$\exists$I, 7
9	$\quad\quad \exists z(Rab \wedge Rbz)$	$\exists$E, 5, 6–8
10	$\quad\quad \exists y\exists z(Ray \wedge Ryz)$	$\exists$I, 9
11	$\quad \exists y\exists z(Ray \wedge Ryz)$	$\exists$E, 3, 4–10
12	$\quad \exists x\exists y\exists z(Rxy \wedge Ryz)$	$\exists$I, 11
13	$\quad \exists x\exists y\exists z(Rxy \wedge Ryz)$	$\exists$E, 2, 3–12

(v) $\exists x(Fx \land \forall y(Fy \to y = x)), \forall x(Fx \leftrightarrow Gx) \vdash \exists x(Gx \land \forall y(Gy \to y = x))$

1	$\exists x(Fx \land \forall y(Fy \to y = x))$	
2	$\forall x(Fx \leftrightarrow Gx)$	
3	$\quad Fa \land \forall y(Fy \to y = a)$	
4	$\quad \forall y(Fy \to y = a)$	$\land$E, 3
5	$\quad Fa$	$\land$E, 3
6	$\quad Fa \leftrightarrow Ga$	$\forall$E, 2
7	$\quad Ga$	$\bot$E, 6, 5
8	$\qquad Gb$	
9	$\qquad Fb \leftrightarrow Gb$	$\forall$E, 2
10	$\qquad Fb$	$\bot$E, 9, 8
11	$\qquad Fb \to b = a$	$\forall$E, 4
12	$\qquad b = a$	$\to$E, 11, 10
13	$\quad Gb \to b = a$	$\to$I, 8–12
14	$\quad \forall y(Gy \to y = a)$	$\forall$I, 13
15	$\quad Ga \land \forall y(Gy \to y = a)$	$\land$I, 7, 14
16	$\quad \exists x(Gx \land \forall y(Gy \to y = x))$	$\exists$I, 15
17	$\exists x(Gx \land \forall y(Gy \to y = x))$	$\exists$E, 1, 3–16

2. Attempt all parts of this question.

(a) Using the following symbolization key

domain: the Muppets
Fx: x is a frog
Jx: x tells jokes
Sx: x is a singer
Fxy: x is funnier than y
Lxy: x loves y
g: Gonzo
k: Kermit
m: Miss Piggy

symbolize each of the following sentences as best you can in FOL, commenting on any difficulties or limitations you encounter [50]

(i) Miss Piggy loves Kermit unless Gonzo is funnier than Kermit.

$Lmk \vee Fgk$

Comment: this assumes that the correct rendering of 'unless' is $\vee$.

(ii) The frog whom Miss Piggy loves also tells jokes.

$\exists x(Fx \wedge Kmx \wedge \forall y((Fy \wedge Kmy) \rightarrow y = x) \wedge Jx)$

Comment: this assumes Russell's theory of descriptions.

(iii) Precisely two singers are loved by Kermit.

$\exists x \exists y(Sx \wedge Sy \wedge Lkx \wedge Lky \wedge \neg x = y \wedge \forall z((Sz \wedge Lkz) \rightarrow (z = x \vee z = y)))$

Comment: Given the domain, this 'says' that precisely two singers *who are also Muppets* are loved by Kermit.

(iv) There are at most three singers, who tell jokes, and none of them are frogs.

$\forall w \forall x \forall y \forall z((Sw \wedge Sx \wedge Sy \wedge Sz) \rightarrow ((w = x \vee w = y \vee w = z \vee x = y \vee x = z \vee y = z) \wedge Jw \wedge Jx \wedge Jy \wedge Jz \wedge \neg Fw \wedge \neg Fx \wedge \neg Fy \wedge \neg Fz))$

Comment: Again, we've expressed that there are at most three singers *who are Muppets*.

(v) A frog's jokes are always funnier than a singer's.

$\forall x \forall y((Fx \wedge Sy \wedge Jx \wedge Jy) \rightarrow Fxy)$

Comment: We've captured 'every joke-telling frog is funnier than every joke-telling singer'. There are at least three limitations here. First, the original is ambiguous between 'there is some frog and some singer such that the jokes of the first are funnier than those of the second' and 'every frog has funnier jokes than every singer'. Second, we don't have the resources here to compare the funniness of *jokes*, only Muppets. Third, we cannot represent the 'always'.

(vi) Both singers and frogs tell jokes

$\exists x \exists y(Sx \wedge Fy \wedge Jx \wedge Jy)$

Comment: The original is multiply ambiguous. It could be read in at least the following ways: 'there is some singer who tells jokes and some frog who tells jokes' or 'all singers and all frogs tell jokes' or 'there is some particular pair of singers who tell jokes and some particular pair of frogs who tell jokes'.

(vii) All frogs love Miss Piggy, and some love Gonzo too.

$\forall x(Fx \to Lxm) \wedge \exists x(Fx \wedge Lxg)$

Comment: The second conjunct of the original is ambiguous between 'some Muppets love Gonzo too' and 'some *frogs* love Gonzo too'.

(viii) Given that Kermit tells jokes, some frog must tell jokes.

$\exists x(Fx \wedge Jx) \to Jk$

Comment: There are two problems here. First, we don't have a canonical symbolization for 'given that'. Let's say that 'A given that B' gets symbolized as '$B \to A$'. Second, the original contains the modal expression 'must', which we cannot represent in FOL.

(ix) Some frog and some singer are exactly as funny as Gonzo.

$\exists x\exists y(Fx \wedge Sy \wedge \neg Fxg \wedge \neg Fyg \wedge \neg Fgx \wedge \neg Fgy)$

Comment: We can't directly represent 'x is exactly as funny as y' so the best we can do is represent 'x isn't funnier than y and y isn't funnier than x'.

(x) Three singing frogs are funnier than Miss Piggy.

$\exists x\exists y\exists z(Fx \wedge Sx \wedge Fy \wedge Sy \wedge Fz \wedge Sz \wedge \neg x = y \wedge \neg x = z \wedge \neg y = z \wedge Fxm \wedge Fym \wedge Fzm)$

Comment: The original is ambiguous between 'there are at least three' and 'there are exactly three'. We've gone with the former.

(b) Show that each of the following claims is true. [40]

(i) $\forall y(\exists xRyx \wedge \exists xQyx) \not\models \forall y\exists x(Ryx \wedge Qyx)$

$D = \{1, 2, 3\}$
$|R| = \{\langle 1, 2\rangle, \langle 2, 1\rangle\}$
$|Q| = \{\langle 1, 3\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 2, 3\rangle\}$

(ii) $\exists x(Px \to Qx) \not\models \exists xPx \to \exists xQx$

$D = \{1, 2\}$
$|P| = \{2\}$
$|Q| = \{\}$

(iii) $\forall x(Px \to \exists y(Rxy \land Ryx)), \neg\exists x Rxx \not\models \forall x \neg Px$

$D = \{1, 2\}$
$|P| = \{1\}$
$|R| = \{\langle 1, 2\rangle, \langle 2, 1\rangle\}$

(iv) $\exists x \forall y(Fy \leftrightarrow x = y), \exists x \forall y(Gy \leftrightarrow x = y) \not\models \exists x \exists y(\neg x = y \land \forall z((Fz \lor Gz) \leftrightarrow (x = z \lor y = z)))$

$D = \{1\}$
$|F| = \{1\}$
$|G| = \{1\}$

(c) Explain the differences between what is symbolized by '$\vdash$' and '$\models$'. [10]

'$\vdash$' expresses syntactic or proof-theoretic consequence: '$\Gamma \vdash \phi$' means that there is a proof of ϕ with no open assumptions but Γ using only the rules of a given deductive system. '$\models$' expresses semantic or model-theoretic consequence: '$\Gamma \models \phi$' means that every model of Γ is a model of ϕ. The two notions can be related by soundness and completeness.

3. Explain the axiom of extensionality. Let R be the set of numbers $\{1, 2, 3, 4\}$, let S be the set $\{3, 4, 5, 6\}$ and let T be $\{5, 6, 7, 8\}$. Write down the elements of the following sets: [20]

Extensionality: $\forall X \forall Y(X = Y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$

(i) $R \cup T = \{1, 2, 3, 4, 5, 6, 7, 8\}$
(ii) $R \cap (S \cup T) = \{3, 4\}$
(iii) $\mathcal{P}(R \cap S) = \{\{3\}, \{4\}, \{3, 4\}, \phi\}$
(iv) $\mathcal{P}(R) \cap \mathcal{P}(S) = \{\{3\}, \{4\}, \{3, 4\}, \phi$
(v) The set y of all elements of S that exceed some element of $S - T$ has no members.

(b) Define reflexivity, symmetry and transitivity. For each of the following relations, say which of these properties it possesses (on the domain of presently living people):

A relation, R, is reflexive with respect to a given domain if, and only if, $\forall x Rxx$.
A relation, R, is symmetric with respect to a given domain if, and only if, $\forall x \forall y(Rxy \to Ryx)$.
A relation, R, is transitive with respect to a given domain if, and only if, $\forall x \forall y \forall z((Rxy \land Ryz) \to Rxz)$.

(i) x is at least one inch taller than y

not-R, not-S, T

(ii) x is at most one inch taller than y

R, not-S, not-T

(iii) x is taller than y iff y is taller than x

R, S, T

(iv) x is exactly 80 years older than y

not-R, not-S, T

(v) x is a brother of y

not-R, not-S, not-T

(vi) x and y are brothers

not-R, S, not-T

(vii) x and y have the same first name or the same last name

R, S, not-T

(viii) x owns a helicopter iff y does

R, not-S, T

(ix) x is David Hasselhoff and y is not

not-R, not-S, T

(x) Most people prefer x to y

not-R, not-S, not-T

4. (a) Write down the probability axioms and the definition of conditional probability. [10]

Probability Axioms:

$Pr(V) = 1$, where V is our sample space (set of possible outcomes).
$Pr(X) \geq 0$ where $X \in \mathcal{P}(V)$
if $X \cap Y = \emptyset$ then $Pr(X \cup Y) = Pr(X) + Pr(Y)$

$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$

(b) I draw three cards at random without replacement from a standard pack of cards. What is the probability:

(i) That the first two are aces? [5]

$$Pr(F2A) = \frac{1}{13} + \frac{3}{51} = \frac{1}{221}$$

(ii) That the first two are aces given that the first is an ace? [10]

$$Pr(F2A|FA) = \frac{Pr(F2A) \cap Pr(FA)}{Pr(FA)} = \frac{Pr(F2A)}{Pr(FA)} = \frac{\frac{1}{221}}{\frac{1}{13}} = \frac{1}{17}$$

(iii) That the first two are aces given that the third is the King of Hearts? [15]

$$Pr(F2A|3KH) = \frac{Pr(F2A \cap 3KH)}{Pr(3KH)} = \frac{\frac{1}{221}x\frac{1}{50}}{\frac{51}{52}x\frac{50}{51}x\frac{1}{50}} = \frac{2}{425}$$

(c) I have 2 bags. One of them contains 50 red balls and 50 black balls. The other contains 60 red balls and 40 black balls. I cannot tell which is which. I choose a bag at random and draw a ball from it (without putting it back). It is red. Then I draw another ball from the same bag. What is the probability that this second ball is red? [30]

$Pr(2R|1R) = \frac{Pr(2R \cap 1R)}{Pr(1R)}$
$Pr(2R \cap 1R) = (\frac{1}{2}x\frac{1}{2}x\frac{49}{99}) + (\frac{1}{2}x\frac{3}{5}x\frac{59}{99}) = \frac{599}{1980}$
$Pr(1R) = (\frac{1}{2}x\frac{1}{2}) + (\frac{1}{2}x\frac{3}{5}) = \frac{11}{20}$
$Pr(2R|1R) = \frac{\frac{599}{1980}}{\frac{11}{20}} = \frac{599}{1089}$

(d) Four balls are placed in a bowl. One is green, one is black and the other two are yellow. The bowl is shaken and someone draws two balls from the bowl. He looks at the two balls and announces that at least one of them is yellow. What is the probability that the other ball he has drawn out is also yellow? [30]

$Pr(2Y|1Y) = \frac{Pr(2Y) \cap Pr(1Y)}{Pr(1Y)}$
$Pr(1Y) = (\frac{1}{2}x\frac{2}{3}) + (\frac{1}{2}x\frac{2}{3}) + (\frac{1}{2}x\frac{1}{3}) = \frac{5}{6}$
$Pr(2Y \cap 1Y) = \frac{1}{2}x\frac{1}{3} = \frac{1}{6}$
$Pr(2Y|1Y) = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5}$