Part IA Paper 5: Formal Methods*

Easter term 2024

Section A

1. Which of the following three statements is true, and which is false? Briefly explain your answers.
   (a) The maximum number of predicates you can form from the following sentence is 3: 'Scott waved to Ramona.' (3)
   (b) Premises tautologically entail a conclusion if and only if there is some valuation according to which all the premises are true and the conclusion false. (3)
   (c) For any TFL sentences **A** and **B**, if **A** tautologically entails **B**, then ¬**A** tautologically entails ¬**B**. (3)

Solution:

(a) False. Here are 5:

- $—_1$ waved to Ramona.
- Scott waved to $—_1$.
- $—_1$ waved to $—_2$.
- $—_2$ waved to $—_1$.
- $—_1$ waved to $—_1$.

(b) False. Rather: Premises tautologically entail a conclusion if and only if there is *no* valuation according to which all the premises are true and the conclusion false.

(c) False. For example, $A \wedge B$ tautologically entails B, but $\neg(A \wedge B)$ does not entail $\neg B$ (consider a valuation in which A is false but B is true).

Total for Question 1: 9

2. Here is an interpretation:

- *Domain*: All books
- *Names*: m: Middlemarch, u: Ulysses, c: The Very Hungry Caterpillar
- *Predicates*: L: $—_1$ is longer than $—_2$; B: $—_1$ is better than $—_2$; W: $—_1$ is widely read

Now provide symbolisations in FOL of the following English sentences:

(a) Middlemarch is better than Ulysses, and longer too. (1)
(b) The Very Hungry Caterpillar is shorter than both Middlemarch and Ulysses, but is either better than both or widely read. (2)

*This is a modified version of the exam paper, to correct the errors (i)–(iii) noted in the Examiners' Report for this paper; an assumption has also been added to question 8(a)ii.

(c) If The Very Hungry Caterpillar is the best book, then any longer book is worse. (2)

(d) Every book longer than Middlemarch is worse than Ulysses. (2)

(e) At least one widely read book is better than some book that is not widely read; and moreover, better than any book longer than Middlemarch. (2)

Solution:

(a) $Bmu \land Lmu$

(b) $Lmc \land Luc \land ((Bcm \land Bcu) \lor Wc)$

(c) $\neg\exists x Bxc \to \forall x(Lxc \to Bcx)$

(d) $\forall x(Lxm \to Bux)$

(e) $\exists x \exists y(Wx \land \neg Wy \land Bxy \land \forall z(Lzm \to Bxz))$

Total for Question 2: 9

3. Prove the following using the natural deduction system in *forallx:Cambridge*.

(a) $(A \to C), (B \to C) \vdash (A \lor B) \to C$ (2)

(b) $\forall x \exists y(Fx \to Gy) \vdash (\forall x Fx \to \exists y Gy)$ (3)

(c) $P \leftrightarrow (Q \land \neg R), (R \lor P) \to \neg Q \vdash P \lor \neg Q$ (4)

Solution:

(a)

1	$\mid A \to C$	
2	$\mid B \to C$	
3	$\mid\ \mid A \lor B$	
4	$\mid\ \mid\ \mid A$	
5	$\mid\ \mid\ \mid C$	$\to$E, 1, 4
6	$\mid\ \mid\ \mid B$	
7	$\mid\ \mid\ \mid C$	$\to$E, 2, 6
8	$\mid\ \mid C$	$\lor$E, 3, 4–5, 6–7
9	$\mid (A \lor B) \to C$	$\to$I, 3–8

(b)

1	$\forall x \exists y(Fx \to Gy)$	
2	$\exists y(Fa \to Gy)$	$\forall$E, 1
3	$\quad Fa \to Gb$	
4	$\quad\quad \forall x Fx$	
5	$\quad\quad Fa$	$\forall$E, 4
6	$\quad\quad Gb$	$\to$E, 3, 5
7	$\quad\quad \exists y Gy$	$\exists$I, 6
8	$\quad \forall x Fx \to \exists y Gy$	$\to$I, 4–7
9	$\forall x Fx \to \exists y Gy$	$\exists$E, 2, 3–8

(c)

1	$P \leftrightarrow (Q \land \neg R)$	
2	$(R \lor P) \to \neg Q$	
3	$\quad P$	
4	$\quad P \lor \neg Q$	$\lor$I, 3
5	$\quad \neg P$	
6	$\quad\quad \neg\neg Q$	
7	$\quad\quad \neg(R \lor P)$	MT, 2, 6
8	$\quad\quad \neg R \land \neg P$	DeM, 7
9	$\quad\quad \neg R$	$\land$E, 8
10	$\quad\quad Q \land \neg R$	$\land$I, 6, 9
11	$\quad\quad P$	$\bot$E, 1, 10
12	$\quad\quad \bot$	$\neg$E, 5, 11
13	$\quad \neg\neg\neg Q$	$\neg$I, 6–12
14	$\quad \neg Q$	DNE, 13
15	$\quad P \lor \neg Q$	$\lor$I, 14
16	$P \lor \neg Q$	TND, 4, 15

Total for Question 3: 9

4. Let $A = \{1, 2, 3\}$, $B = \{1, \{2, 3\}\}$, and $C = \{3, 4\}$. Explicitly give the members of the following.
   (a) $A \cup B$ (1)
   (b) $A \cap C$ (1)
   (c) A^2 (2)

(d) $(A - B) \times C$ (2)

(e) $\mathcal{P}(B) \cup (\mathcal{P}(C) - \mathcal{P}(A))$ (3)

Solution:

(a) $\{1, 2, 3, \{2, 3\}\}$

(b) $\{3\}$

(c) $\{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 3, 3\rangle\}$

(d) $\{\langle 2, 3\rangle, \langle 3, 3\rangle, \langle 2, 4\rangle, \langle 3, 4\rangle\}$

(e) $\{\{4\}, \{3, 4\}, \{1\}, \{\{2, 3\}\}, \{1, \{2, 3\}\}, \varnothing\}$

Answers without the outermost braces were also accepted.

Total for Question 4: 9

5. A bag contains two cubes, two spheres, and two pyramids. One cube is blue, the other is red. One sphere is yellow, the other is blue. One pyramid is yellow and the other is red. If two shapes are randomly drawn from the bag, one after the other and without replacement, calculate the probability of

(a) at least one shape being either red or a cube. (2)

(b) both the first shape being neither a pyramid nor blue and the second being neither yellow nor a cube. (3)

(c) the second shape being a pyramid, given that the first was red. (4)

Solution:

(a) 4/5. (Only 6 outcomes out of 30 where both shapes are not red and not a cube. $1 - 6/30 = 12/15 = 4/5$.)

(b) 2/15. (First shape must be either red cube or yellow sphere, second must be either blue sphere or red pyramid. Four outcomes.)

(c) 3/10. (Probability that the first shape is red and the second is a pyramid is 1/10 (three outcomes); probability that the first shape is red is 1/3.)

Total for Question 5: 9

Section B

6. A rule of inference is said to be *sound* if it only permits valid inferences. For example, consider the rule for $\wedge E$:

m	$\mathbf{A} \wedge \mathbf{B}$			m	$\mathbf{A} \wedge \mathbf{B}$		
	$\mathbf{A}$	$\wedge$E, m			$\mathbf{B}$	$\wedge$E, m	

This is shown to be sound by the following pair of truth-tables:

A	**B**	$\mathbf{A} \wedge \mathbf{B}$	**A**
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

A	**B**	$\mathbf{A} \wedge \mathbf{B}$	**B**
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	F

(a) For each of the following rules of inference, use truth-tables to show its soundness.

i. $\wedge$I (3)

ii. $\vee$I (3)

iii. $\rightarrow$E (3)

iv. $\leftrightarrow$E (3)

v. $\neg$E (3)

vi. X (3)

You may use the fact that $\bot$ takes the value F in every valuation.

Solution:

i.

A	**B**	**A**	**B**	$\mathbf{A} \wedge \mathbf{B}$
T	T	T	T	T
T	F	T	F	F
F	T	F	T	F
F	F	F	F	F

ii.

A	**B**	**A**	$\mathbf{A} \vee \mathbf{B}$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	F	F

A	**B**	**B**	$\mathbf{A} \vee \mathbf{B}$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	F	F

iii.

A	**B**	**A**	$\mathbf{A} \rightarrow \mathbf{B}$	**B**
T	T	T	T	T
T	F	T	F	F
F	T	F	T	T
F	F	F	T	F

iv.

A	**B**	**A**	$\mathbf{A} \leftrightarrow \mathbf{B}$	**B**
T	T	T	T	T
T	F	T	F	F
F	T	F	F	T
F	F	F	T	F

A	**B**	**B**	**A** ↔ **B**	**A**
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	F	T	F

v.

A	**A**	¬**A**	⊥
T	T	F	F
F	F	T	F

vi.

A	⊥	**A**
T	F	T
F	F	F

(b) Can we use a truth-table to show the soundness of →I? Briefly explain why or why not. (2)

Solution: No: the rule →I presupposes not just that we are given certain sentences, but that we are given certain subproofs. This goes beyond what can be shown in a single truth-table.

Total for Question 6: 20

7. (a) Use the natural deduction system in *forallx:Cambridge* to prove that identity is an equivalence relation. (5)

Solution:

1	$a = a$	$=$I
2	$\forall x(x = x)$	$\forall$I, 1

1	$\quad \mid\ a = b$	
2	$\quad \mid\ a = a$	$=$I
3	$\quad \mid\ b = a$	$=$E, 1, 2
4	$a = b \to b = a$	$\to$I, 1–3
5	$\forall y(a = y \to y = a)$	$\forall$I, 4
6	$\forall x \forall y(x = y \to y = x)$	$\forall$I, 5

1	$\quad \mid\ a = b$	
2	$\quad \mid \quad \mid\ b = c$	
3	$\quad \mid \quad \mid\ a = c$	$=$E, 1, 2
4	$\quad \mid\ b = c \to a = c$	$\to$I, 2–3
5	$a = b \to (b = c \to a = c)$	$\to$I, 1–4
6	$\forall z(a = b \to (b = z \to a = z))$	$\forall$I, 5
7	$\forall y \forall z(a = y \to (y = z \to a = z))$	$\forall$I, 6
8	$\forall x \forall y \forall z(x = y \to (y = z \to x = z))$	$\forall$I, 7

(b) Let R be a two-place predicate. We let the two-place predicate $\sim$ of *R-indiscernibility* be defined by the following sentence of FOL: (6)

$$\forall x \forall y\big(x \sim y \leftrightarrow \forall z\big((Rxz \leftrightarrow Ryz) \land (Rzx \leftrightarrow Rzy)\big)\big) \qquad (*)$$

Show that the definition (*) entails that R-indiscernibility is an equivalence relation. You may do so by reasoning about all interpretations, or by using the natural deduction system in *forallx:Cambridge*.

Solution: Let I be an interpretation in which (*) holds.

First, consider any objects a and b in the domain of I, let I^+ be any interpretation that extends I by assigning a to the name m, and let I^{++} an any interpretation that extends I^+ by assigning b to the name n. Whatever the extension of R, the sentences $Rmn \leftrightarrow Rmn$ and $Rnm \leftrightarrow Rnm$ will be true in I^{++}. So $\forall z((Rmz \leftrightarrow Rmz) \land (Rzm \leftrightarrow Rzm))$ will be true in I^+. So $m \sim m$ will be true in I^+, and hence $\langle a, \rangle$ is in the extension of $\sim$ in I.

Second, consider any objects a and b in the domain of I such that $\langle a, b\rangle$ is in the extension of $\sim$. Let I^+ be any interpretation that extends I by assigning m and n to a and b respectively. Then $\forall z((Rmz \leftrightarrow Rnz) \land (Rzm \leftrightarrow Rzn))$ is true in I^+, from which it follows that $\forall z((Rnz \leftrightarrow Rmz) \land (Rzn \leftrightarrow Rzm))$ is true in I^+; hence, $n \sim m$ is true in I^+, and so $\langle b, a\rangle$ is in the extension of $\sim$ in I.

Finally, consider any objects a, b, c in the domain of I such that $\langle a, b\rangle$ and $\langle b, c\rangle$ are in the extension of $\sim$. Let I^+ be any interpretation that extends I by assigning m, n, p to a, b, c respectively. Then we know that the following sentence is true in I^+:

$$\forall z((Rmz \leftrightarrow Rnz) \wedge (Rzm \leftrightarrow Rzn) \wedge (Rnz \leftrightarrow Rpz) \wedge (Rzn \leftrightarrow Rzp))$$

So for any interpretation I^{++} that extends I^+ by assigning the name q to some object,

$$(Rmq \leftrightarrow Rnq) \wedge (Rqm \leftrightarrow Rqn) \wedge (Rnq \leftrightarrow Rpq) \wedge (Rqn \leftrightarrow Rqp)$$

But it follows that in I^{++}, the following sentence is true:

$$(Rmq \leftrightarrow Rpq) \wedge (Rqm \leftrightarrow Rqp)$$

Hence, in I^+, it is true that $\forall z((Rmz \leftrightarrow Rpz) \wedge (Rzm \leftrightarrow Rzp))$; and hence, that $m \sim p$. So in I, $\langle a, c\rangle$ is in the extension of $\sim$.

(c) Let J be an interpretation in which (*) is true, and let a and b be R-indiscernible objects in the domain of J. Show that the extension of R in J must either include all of $\langle a, b\rangle$, $\langle b, a\rangle$, $\langle a, a\rangle$, and $\langle b, b\rangle$, or none of them. (3)

Solution: Let J^+ be any interpretation that extends J by assigning the names m and n to a and b respectively. Then $m \sim n$ is true in J^+, so the following sentences are true in J^+:

$$Rmn \leftrightarrow Rnn$$
$$Rnm \leftrightarrow Rnn$$
$$Rmm \leftrightarrow Rnm$$
$$Rmm \leftrightarrow Rmn$$

Thus, if Rmn is true in J^+, then Rmm and Rnn are true, and hence Rnm is true; similarly if any one of Rmm, Rnn, or Rnm are true, so are the other three. So if any of $\langle a, b\rangle$, $\langle b, a\rangle$, $\langle a, a\rangle$, or $\langle b, b\rangle$ are in the extension of R then all the others are.

(d) Given two two-place predicates S and T, we let the two-place predicate I of their *intersection* be defined by the sentence (6)

$$\forall x \forall y(Ixy \leftrightarrow (Sxy \wedge Txy))$$

and we let the two-place predicate U of their *union* be defined by the sentence

$$\forall x \forall y(Uxy \leftrightarrow (Sxy \vee Txy))$$

Show that if S and T are equivalence relations, then I is an equivalence relation, but U need not be.

Solution: To show that the intersection of equivalence relations must be an equivalence relation, suppose that S and T are equivalence relations on some interpretation I. For any object a with the name m, Smm and Tmm, so Imm; hence, I is reflexive. For any objects a and b with the names m and n, if Imn then Smn and Tmn; so Snm and Tnm, hence Inm. So I is symmetric. For any objects a, b, c with the names m, n, p, if Imn and Inp then Smn, Snp, Tmn and Tnp. Hence Smp and Tmp, so Imp. So I is transitive.

To show that the union of equivalence relations need not be an equivalence relation, let the domain consist of 0, 1 and 2. Let the extension of S consist of $\langle 0, 0\rangle$, $\langle 1, 1\rangle$, $\langle 2, 2\rangle$, and $\langle 0, 1\rangle$;

and let the extension of T consist of $\langle 0,0\rangle$, $\langle 1,1\rangle$, $\langle 2,2\rangle$, and $\langle 1,2\rangle$. Then the extension of U is $\langle 0,0\rangle$, $\langle 1,1\rangle$, $\langle 2,2\rangle$, $\langle 0,1\rangle$, and $\langle 1,2\rangle$. So U is not transitive: $\langle 0,1\rangle$ and $\langle 1,2\rangle$ are in its extension, but $\langle 0,2\rangle$ is not.

Total for Question 7: 20

8. (a) Say that a binary relation R is *dense* just in case $\forall x \forall y(Rxy \rightarrow \exists z(Rxz \wedge Rzy))$. Let C be the set of all cats, and let D be the set of all dogs. Say whether the following are reflexive, symmetric, transitive, and/or dense on the domain $C \cup D$.

i. x is a cat and y is a dog. (4)

ii. x has at least as many legs as y. (Assuming that some but not all cats and dogs have four legs.) (4)

iii. x has a tail or y is a cat. (Assuming that there are some cats and dogs with tails, and some cats and dogs with no tails.) (4)

Solution:

i. Not reflexive, not symmetric, transitive, not dense.

ii. Reflexive, not symmetric, transitive, dense.

iii. Not reflexive, not symmetric, not transitive, dense.

(b) Let R' be the *inverse* of binary relation R just in case $\forall x \forall y(Rxy \leftrightarrow R'yx)$. For each of the following explain your answer, and provide an example if appropriate.

i. Can there be an R which is both reflexive and not dense? (2)

ii. Can there be an R which is both transitive and not dense? (2)

iii. Can there be an R which is both symmetric and dense, but whose inverse R' is not dense? (2)

iv. Can there be an R which is transitive, but whose inverse R' is not transitive? (2)

Solution:

i. No. Suppose D is some arbitrary domain. For arbitrary $a, b \in D$, suppose Rab. If R is reflexive, Rbb. Thus, Rab and Rbb. Thus, if Rab, then $\exists z(Raz \wedge Rzb)$. Since $a, b \in D$ were arbitrary, generalise: $\forall x \forall y(Rxy \rightarrow \exists z(Rxz \wedge Rzy))$. Thus, R, if reflexive, is dense.

ii. Yes. Let the domain consist of 1 and 2, and let R be the "strictly greater than" relation. R is transitive but not dense: $2 > 1$ but there is no z such that $2 > z$ and $z > 1$.

iii. No. Suppose R is some arbitrary symmetric and dense binary relation and D an arbitrary domain. Suppose for arbitrary $a, b \in D$, $R'ab$. Thus, Rba and Rab from inverse and symmetry. Given that R is dense, $\exists z(Raz \wedge Rzb)$. R is symmetric. Thus, $\exists z(Rza \wedge Rbz)$. From inverse, $\exists z(R'az \wedge R'zb)$. Since $a, b \in D$ were arbitrary, it follows from supposition that $\forall x \forall y(R'xy \rightarrow \exists z(R'xz \wedge R'zy))$.

Alternatively: if R is symmetric, then $R' = R$. So if R is dense, so is R'.

iv. No. Suppose that $R'ab$ and $R'bc$. Then Rba and Rcb. So Rca (since R is transitive). So $R'ac$.

Total for Question 8: 20

9. The Buttery sell three types of doughnut: blue-iced, red-iced, and green-iced. In an effort to curb doughnut consumption, The Buttery begins adding a chemical C to the icing. If someone consumes at least 4mg of C, they experience the taste of sour milk. Answer the following, explaining your answer. (For all questions, you may assume that people only taste sour milk as a result of consuming at least 4mg of C, and that no-one has consumed any C prior to the question.)

   (a) Alice randomly purchases and eats two doughnuts from a selection of five comprised of two blue-iced, two red-iced, and one green-iced. She selects one after another without replacement. If blue ones contain 3mg of C, red ones contain 2mg of C, and green ones contain 1mg of C, calculate:

      i. the probability that Alice's selection consists of one blue doughnut and one green doughnut. (2)

      ii. the probability that Alice experiences the taste of sour milk. (3)

      iii. the probability that Alice experiences the taste of sour milk, given that her selection contains exactly one red doughnut. (2)

Solution:

i. 1/5. The probability that Alice selects a blue doughnut then a green doughnut is $2/5 \times 1/4 = 1/10$. Similarly, the probability that she selects a green doughnut then a blue doughnut is $1/5 \times 2/4 = 1/10$. So the probability that she selects a red and a blue doughnut (in either order) is 1/5.

ii. 4/5. The only (possible) selections containing less than 4mg of C are those consisting of a red doughnut and a green doughnut. The probability that Alice selects a red doughnut then a green doughnut is $2/5 \times 1/4 = 1/10$; the probability that she selects a green doughnut then a red doughnut is $1/5 \times 2/4 = 1/10$. So the probability that she consumes less than 4mg of C is 1/5—and hence the probability that she consumes at least 4mg is 4/5.

iii. Alice selects exactly one red if either (i) she selects red first and then doesn't select red or (ii) she doesn't select red first and then selects red. $Pr(\text{i}) = (2/5 \times 3/4)$. $Pr(\text{ii}) = (3/5 \times 2/4)$. The probability that Alice selects exactly one red is $(3/10 + 3/10) = 3/5$. Alice tastes sour milk and consumes exactly one red doughnut if and only if her selection consists of one red and one blue doughnut. By similar reasoning to the above, the probability of this is 2/5. So, the probability that Alice tastes sour milk given that she consumes exactly one red doughnut is $(2/5)/(3/5) = 2/3$.

   Alternatively: given that Alice consumes exactly one red doughnut, she will taste sour milk if and only if her other doughnut is blue; and the only non-red doughnuts are the two blue and one green. So the probability that she tastes sour milk is the probability that she gets a blue from that selection, i.e. 2/3.

   (b) A device is invented to detect C in doughnuts. The probability that the device beeps in the presence of a doughnut, given that the doughnut contains C, is 9/10; the probability that the device beeps, given that the doughnut does not contain C, is 1/2. (4)

   The buttery prepares a selection five pink-iced doughnuts, three of which each contain 4mg of C and two of which contain no C. Bob randomly chooses a doughnut from this selection, and tests it with the device. What is the probability that the doughnut does *not* contain any C, given that the device beeps?

Solution: $P(B) = P(B|C)P(C) + P(B|\neg C)P(\neg C) = \frac{9}{10} \times \frac{3}{5} + \frac{1}{2} \times \frac{2}{5} = \frac{37}{50}$

$P(C|B) = \frac{P(B|C)P(C)}{P(B)} = \frac{27}{37}$

$P(\neg C|B) = 1 - P(C|B) = \frac{10}{37}$

   (c) Consider a new selection of doughnuts comprised of two blue-iced, two red-iced, and one green-iced. Blue ones now contain 2mg of C, red ones contain 1mg of C, and green ones contain 0.5mg of (5)

C. Claire randomly picks three from the new selection, one after another without replacement. Calculate the probability that Claire's selection contains enough C to give the taste of sour milk.

Solution: 1/2. Any three containing enough C must consist of either (a) two red, one blue, (b) two blue, one red, or (c) two blue, one green. The selection satisfies (a) if either:

(i) Blue is selected first, then red, and then red. $Pr(\text{i}) = (2/5 \times 1/2 \times 1/3) = 1/15$

(ii) Red is selected first, then blue, and then red. $Pr(\text{ii}) = (2/5 \times 1/2 \times 1/3) = 1/15$

(iii) Red is selected first, then red, and then blue. $Pr(\text{iii}) = (2/5 \times 1/4 \times 2/3) = 1/15$

Thus, $Pr(\text{a}) = 1/5$. The selection satisfies (b) if either:

(i) Blue is selected first, then blue, and then red. $Pr(\text{i}) = (2/5 \times 1/4 \times 2/3) = 1/15$

(ii) Blue is selected first, then red, and then blue. $Pr(\text{ii}) = (2/5 \times 1/2 \times 1/3) = 1/15$

(iii) Red is selected first, then blue, and then blue. $Pr(\text{iii}) = (2/5 \times 1/2 \times 1/3) = 1/15$

Thus, $Pr(\text{b}) = 1/5$. The selection satisfies (c) if either:

(i) Blue is selected first, then blue, and then green. $Pr(\text{i}) = (2/5 \times 1/4 \times 1/3) = 1/30$

(ii) Blue is selected first, then green, and then blue. $Pr(\text{ii}) = (2/5 \times 1/4 \times 1/3) = 1/30$

(iii) Green is selected first, then blue, and then blue $Pr(\text{iiii}) = (1/5 \times 1/2 \times 1/3) = 1/30$

Thus, $Pr(\text{c}) = 1/10$. Thus, $Pr(a \cup b \cup c) = 1/2$.

(d) Dave and Emma play the following game. Dave randomly picks three doughnuts one after another without replacement from the selection described in (c), and eats the first and third doughnuts only. Emma tosses an unfair coin. Dave wins if and only if the coin lands heads and he does *not* experience the taste of sour milk. If the probability that Dave wins is 16/25, calculate the probability that the coin toss lands heads. (4)

Solution: 3/5. The first and third doughnut contain enough C only if both are blue. Thus, the probability that Dave tastes sour milk after eating the first and third is $(2/5 \times 3/4 \times 1/3) = 1/10$. Thus, the probability that Dave *does not* taste sour milk after eating the first and third is 9/10. The probability that Dave wins is $(1 - 16/25) = 9/25$. Dave wins (D) if and only if he doesn't consume enough C to taste sour milk ($\neg C$) and the coin doesn't lands heads (H). Thus:

$$Pr(D) = Pr(\neg C) \times (1 - Pr(H)) = 9/25$$

Thus:

$$Pr(H) = 1 - (\frac{9/25}{9/10}) = 3/5$$

Total for Question 9: 20

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