Part IA Paper 5: Formal Methods*

Easter term 2024

Section A

1. Which of the following three statements is true, and which is false? Briefly explain your answers.
   (a) The maximum number of predicates you can form from the following sentence is 3: 'Scott waved to Ramona.'
   (b) Premises tautologically entail a conclusion if and only if there is some valuation according to which all the premises are true and the conclusion false.
   (c) For any TFL sentences **A** and **B**, if **A** tautologically entails **B**, then $\neg$**A** tautologically entails $\neg$**B**.

2. Here is an interpretation:
   - *Domain*: All books
   - *Names*: m: Middlemarch, u: Ulysses, c: The Very Hungry Caterpillar
   - *Predicates*: L: $—_1$ is longer than $—_2$; B: $—_1$ is better than $—_2$; W: $—_1$ is widely read

   Now provide symbolisations in FOL of the following English sentences:
   (a) Middlemarch is better than Ulysses, and longer too.
   (b) The Very Hungry Caterpillar is shorter than both Middlemarch and Ulysses, but is either better than both or widely read.
   (c) If The Very Hungry Caterpillar is the best book, then any longer book is worse.
   (d) Every book longer than Middlemarch is worse than Ulysses.
   (e) At least one widely read book is better than some book that is not widely read; and moreover, better than any book longer than Middlemarch.

3. Prove the following using the natural deduction system in *forallx:Cambridge*.
   (a) $(A \to C), (B \to C) \vdash (A \lor B) \to C$
   (b) $\forall x \exists y(Fx \to Gy) \vdash (\forall x Fx \to \exists y Gy)$
   (c) $P \leftrightarrow (Q \land \neg R), (R \lor P) \to \neg Q \vdash P \lor \neg Q$

4. Let $A = \{1, 2, 3\}$, $B = \{1, \{2, 3\}\}$, and $C = \{3, 4\}$. Explicitly give the members of the following.
   (a) $A \cup B$
   (b) $A \cap C$
   (c) A^2
   (d) $(A - B) \times C$
   (e) $\mathcal{P}(B) \cup (\mathcal{P}(C) - \mathcal{P}(A))$

*This is a modified version of the exam paper, to correct the errors (i)–(iii) noted in the Examiners' Report for this paper; an assumption has also been added to question 8(a)ii.

5. A bag contains two cubes, two spheres, and two pyramids. One cube is blue, the other is red. One sphere is yellow, the other is blue. One pyramid is yellow and the other is red. If two shapes are randomly drawn from the bag, one after the other and without replacement, calculate the probability of

   (a) at least one shape being either red or a cube.

   (b) both the first shape being neither a pyramid nor blue and the second being neither yellow nor a cube.

   (c) the second shape being a pyramid, given that the first was red.

Section B

6. A rule of inference is said to be *sound* if it only permits valid inferences. For example, consider the rule for $\wedge E$:

m	$\mathbf{A} \wedge \mathbf{B}$	
	$\mathbf{A}$	$\wedge$E, m

m	$\mathbf{A} \wedge \mathbf{B}$	
	$\mathbf{B}$	$\wedge$E, m

This is shown to be sound by the following pair of truth-tables:

A	**B**	$\mathbf{A} \wedge \mathbf{B}$	**A**
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

A	**B**	$\mathbf{A} \wedge \mathbf{B}$	**B**
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	F

(a) For each of the following rules of inference, use truth-tables to show its soundness.

i. $\wedge$I

ii. $\vee$I

iii. $\rightarrow$E

iv. $\leftrightarrow$E

v. $\neg$E

vi. X

You may use the fact that $\bot$ takes the value F in every valuation.

(b) Can we use a truth-table to show the soundness of $\rightarrow$I? Briefly explain why or why not.

7. (a) Use the natural deduction system in *forallx:Cambridge* to prove that identity is an equivalence relation.

   (b) Let R be a two-place predicate. We let the two-place predicate $\sim$ of *R-indiscernibility* be defined by the following sentence of FOL:

$$\forall x \forall y \big(x \sim y \leftrightarrow \forall z \big((Rxz \leftrightarrow Ryz) \land (Rzx \leftrightarrow Rzy)\big)\big) \qquad (*)$$

   Show that the definition (*) entails that R-indiscernibility is an equivalence relation. You may do so by reasoning about all interpretations, or by using the natural deduction system in *forallx:Cambridge*.

   (c) Let J be an interpretation in which (*) is true, and let a and b be R-indiscernible objects in the domain of J. Show that the extension of R in J must either include all of $\langle a, b\rangle$, $\langle b, a\rangle$, $\langle a, a\rangle$, and $\langle b, b\rangle$, or none of them.

   (d) Given two two-place predicates S and T, we let the two-place predicate I of their *intersection* be defined by the sentence

$$\forall x \forall y (Ixy \leftrightarrow (Sxy \land Txy))$$

   and we let the two-place predicate U of their *union* be defined by the sentence

$$\forall x \forall y (Uxy \leftrightarrow (Sxy \lor Txy))$$

   Show that if S and T are equivalence relations, then I is an equivalence relation, but U need not be.

8. (a) Say that a binary relation R is *dense* just in case $\forall x \forall y (Rxy \rightarrow \exists z (Rxz \wedge Rzy))$. Let C be the set of all cats, and let D be the set of all dogs. Say whether the following are reflexive, symmetric, transitive, and/or dense on the domain $C \cup D$.

   i. x is a cat and y is a dog.

   ii. x has at least as many legs as y. (Assuming that some but not all cats and dogs have four legs.)

   iii. x has a tail or y is a cat. (Assuming that there are some cats and dogs with tails, and some cats and dogs with no tails.)

   (b) Let R' be the *inverse* of binary relation R just in case $\forall x \forall y (Rxy \leftrightarrow R'yx)$. For each of the following explain your answer, and provide an example if appropriate.

   i. Can there be an R which is both reflexive and not dense?

   ii. Can there be an R which is both transitive and not dense?

   iii. Can there be an R which is both symmetric and dense, but whose inverse R' is not dense?

   iv. Can there be an R which is transitive, but whose inverse R' is not transitive?

9. The Buttery sell three types of doughnut: blue-iced, red-iced, and green-iced. In an effort to curb doughnut consumption, The Buttery begins adding a chemical C to the icing. If someone consumes at least 4mg of C, they experience the taste of sour milk. Answer the following, explaining your answer. (For all questions, you may assume that people only taste sour milk as a result of consuming at least 4mg of C, and that no-one has consumed any C prior to the question.)

   (a) Alice randomly purchases and eats two doughnuts from a selection of five comprised of two blue-iced, two red-iced, and one green-iced. She selects one after another without replacement. If blue ones contain 3mg of C, red ones contain 2mg of C, and green ones contain 1mg of C, calculate:

      i. the probability that Alice's selection consists of one blue doughnut and one green doughnut.

      ii. the probability that Alice experiences the taste of sour milk.

      iii. the probability that Alice experiences the taste of sour milk, given that her selection contains exactly one red doughnut.

   (b) A device is invented to detect C in doughnuts. The probability that the device beeps in the presence of a doughnut, given that the doughnut contains C, is $9/10$; the probability that the device beeps, given that the doughnut does not contain C, is $1/2$.

   The buttery prepares a selection five pink-iced doughnuts, three of which each contain 4mg of C and two of which contain no C. Bob randomly chooses a doughnut from this selection, and tests it with the device. What is the probability that the doughnut does *not* contain any C, given that the device beeps?

   (c) Consider a new selection of doughnuts comprised of two blue-iced, two red-iced, and one green-iced. Blue ones now contain 2mg of C, red ones contain 1mg of C, and green ones contain 0.5mg of C. Claire randomly picks three from the new selection, one after another without replacement. Calculate the probability that Claire's selection contains enough C to give the taste of sour milk.

   (d) Dave and Emma play the following game. Dave randomly picks three doughnuts one after another without replacement from the selection described in (c), and eats the first and third doughnuts only. Emma tosses an unfair coin. Dave wins if and only if the coin lands heads and he does *not* experience the taste of sour milk. If the probability that Dave wins is $16/25$, calculate the probability that the coin toss lands heads.

END OF PAPER