Part IA Paper 5 (Formal Methods)

Easter term 2022

Section A

1. Which of the following claims are true, and which are false? Briefly explain your answers.
   (a) Any jointly tautologically consistent sentences admit of a valuation where all the sentences are false.
   (b) Any jointly tautologically inconsistent sentences will remain tautologically inconsistent if any other sentence is added.
   (c) If an argument's premises are jointly inconsistent with its conclusion, the argument must be invalid.

Solution:

(a) False: that there is a valuation where all the sentences are true does not entail that there is one where they are all false. For example, $(A \vee \neg A)$ is consistent with B, yet there is no valuation rendering both sentences false.

(b) True: if there is no valuation which makes all the sentences true, then that will still be the case with more sentences.

(c) False: if the argument's premises are inconsistent by themselves then the argument is valid, yet the premises and conclusion are jointly inconsistent.

2. Determine whether the following claims are true or false. Where true, prove this by using the natural deduction system in *forallx:Cam*; where false, provide a truth-table.
   (a) $A \wedge \neg B$ is tautologically equivalent to $\neg A \vee B$
   (b) $A \rightarrow (\neg B \rightarrow \neg A)$ is tautologically equivalent to $A \rightarrow B$
   (c) $A \rightarrow (B \leftrightarrow C)$ is tautologically equivalent to $(B \rightarrow C) \wedge (\neg A \vee B \vee \neg C)$

Solution:

(a) False. Any line from the truth-table is sufficient, as the two sentences have different truth-values on every valuation:

A	B	$A \wedge \neg B$	$\neg A \vee B$
T	T	F	T
T	F	T	F
F	T	F	T
F	F	F	T

(b) True. Any pair of valid natural deductions were accepted (note that two are required, to prove equivalence). Example answers:

1	$A \to (\neg B \to \neg A)$	
2	A	
3	$\neg B \to \neg A$	$\to$E, 1, 2
4	$\neg B$	
5	$\neg A$	$\to$E, 3, 4
6	$\bot$	$\neg$E, 2, 5
7	$\neg\neg B$	$\neg$I, 4–6
8	B	DNE, 7
9	$A \to B$	$\to$I, 2–8

1	$A \to B$	
2	A	
3	$\neg B$	
4	$\neg A$	MT, 1, 3
5	$\neg B \to \neg A$	$\to$I, 3–4
6	$A \to (\neg B \to \neg A)$	$\to$I, 2–5

(c) False. The following line of the relevant truth-table suffices to demonstrate this:

A	B	C	$A \to (B \leftrightarrow C)$	$(B \to C) \wedge (\neg A \vee B \vee \neg C)$
F	T	F	T	F

(In fact, this is the only line of the truth-table on which the truth-values of the two sentences diverge.)

3. Use the following symbolisation key for this question:
   - Domain: all species of animals
   - S: $—_1$ is covered in spots
   - M: $—_1$ is a species of monkey
   - H: $—_1$ hunts $—_2$

   Give symbolisations of the following sentences:

   (a) Everything is hunted by something.
   (b) There are exactly two species which are not covered in spots.
   (c) All monkey species are hunters.

   Offer natural-language versions of the following claims:

   (d) $\exists x \forall y (Hxy \to Sy)$
   (e) $\exists x (Mx \land Sx \land \forall y ((My \land Sy) \to y = x))$
   (f) $\forall x ((Sx \land \exists y (My \land Hxy)) \to \exists y (\neg My \land Hxy))$

   Solution:

   (a) $\forall x \exists y (Hyx)$

   (b) $\exists x \exists y (\neg Sx \land \neg Sy \land x \neq y \land \forall z (\neg Sz \to z = x \lor z = y))$

   (c) $\forall x (Mx \to \exists y Hxy)$

   (d) There is a species which only hunts spotted species.

   (e) There is exactly one species of spotted monkey.

   (f) Anything spotted which hunts monkeys, also hunts some non-monkeys.

4. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ then write down the members of the following sets:

   (a) $A \cap B$
   (b) $A \cup B$
   (c) $A - (A - B)$
   (d) $\varnothing - B$
   (e) $A \cap \wp(A)$
   (f) $\wp(\wp(A \cap B))$

   Solution:

   (a) 3

   (b) $1, 2, 3, 4, 5$

   (c) 3

   (d) No members

   (e) No members

   (f) $\varnothing, \{\varnothing\}, \{\{3\}\}, \{\varnothing, \{3\}\}$

   Solutions which specified the sets themselves (rather than the members) were given equal credit.

5. Give examples of relations with the following properties:
   (a) Reflexive and symmetric but not transitive
   (b) Symmetric and transitive but not reflexive
   (c) Transitive and reflexive but not symmetric

Solution: Any relations satisfying the relevant properties were accepted. Relations not satisfying all the relevant properties received a mark for each of the three required properties they exhibited (so, for instance, a relation that was reflexive but not symmetric or transitive would receive two marks for part (a)).

Section B (20 marks each)

1. (a) Consider the interpretation whose domain consists of Vincent, Mary, Paul, and Fanette, all of whom are distinct from one another; and where the predicate 'R' is to be true of (and only of)

$$\langle\text{Vincent, Fanette}\rangle, \langle\text{Paul, Fanette}\rangle, \langle\text{Mary, Vincent}\rangle, \langle\text{Fanette, Paul}\rangle$$

Determine the truth values of each of the following sentences in this interpretation. You do not need to explain your answers.

$$\forall x\forall y\forall z\forall w(x = y \lor x = z \lor x = w \lor y = z \lor y = w \lor z = w) \tag{A}$$

$$\forall x\forall y\forall z(Rxy \to (Ryz \to Rxz)) \tag{B}$$

$$\forall x\exists yRxy \tag{C}$$

$$\forall x\neg Rxx \tag{D}$$

Solution:

(A) False

(B) False

(C) True

(D) True

(b) Demonstrate that the above sentences (A)–(D) are not jointly consistent. You may do so by reasoning about interpretations, or (if you prefer) by the use of the natural deduction system in *forallx:Cam*.

Solution: Consider any interpretation with at most three individuals, in order that (A) be true. Call these a, b and c. Then a must bear R to some individual, in order that (C) be true; and that individual cannot be a itself, otherwise (D) is false. So suppose a bears R to b. By the same reasoning, b must bear R to either a or c. But if it bears R to a, then by (B), a must bear R to itself, contradicting (D). So b bears R to c. Then c must bear R to a or b. If the latter, then by the same reasoning as above, we find that b bears R to itself, contradicting (D). But if the former, then by (B) we get that b bears R to a, which we have already shown to yields a contradiction. So there can be no such interpretation: that is, (A)–(D) are jointly inconsistent.

(c) Show that any three out of the four sentences are jointly consistent: that is, show that
  i. sentences (A), (B), and (C) are consistent;
  ii. sentences (B), (C), and (D) are consistent;
  iii. sentences (A), (B), and (D) are consistent; and
  iv. sentences (A), (C), and (D) are consistent.

Solution: Any interpretations demonstrating the required consistency claims were accepted. An interpretation that made two of the three relevant sentences true received one point. Repeated mistakes (e.g. offering an interpretation with four individuals as making (A) true) were only penalised once. On the other hand, offering the same interpretation for two parts attracted a penalty point, as this would imply that all four sentences are consistent after all.

Example answers:

i. The universal relation over a set with three or fewer members.

ii. The less-than ($<$) relation on the positive whole numbers.

iii. The empty relation over a set with three or fewer members.

iv. A cyclic relation on a set with three or fewer members.

(d) In TFL, one can verify that some sentences are not jointly consistent by considering finitely many valuations (e.g. in the form of a truth-table). Briefly explain why considering finitely many interpretations is not, in general, sufficient to verify that some FOL-sentences are not jointly consistent.

Solution: Because in FOL, there will be infinitely many different interpretations one can offer for any sentence, even though each sentence only contains finitely many letters. By contrast, in TFL, there are only finitely many different valuations of a finite sentence.

2. (a) Prove the following using the natural deduction system in *forallx:Cam*:

   i. $\forall x \forall y(Px \to (Qy \to \neg Rxyx)) \vdash \forall x \forall y(Rxyx \to (Qy \to \neg Px))$

Solution: Any valid natural deduction was accepted. Example answer:

1	$\underline{\forall x \forall y(Px \to (Qy \to \neg Rxyx))}$	
2	$\forall y(Pa \to (Qy \to \neg Raya)$	∀E, 1
3	$Pa \to (Qb \to \neg Raba)$	∀E, 2
4	$\vert\ \underline{Raba}$	
5	$\vert\ \vert\ \underline{Qb}$	
6	$\vert\ \vert\ \vert\ \underline{Pa}$	
7	$\vert\ \vert\ \vert\ Qb \to \neg Raba$	→E, 3, 6
8	$\vert\ \vert\ \vert\ \neg Raba$	→E, 7, 5
9	$\vert\ \vert\ \vert\ \bot$	¬E, 4, 8
10	$\vert\ \vert\ \neg Pa$	¬I, 6–9
11	$\vert\ Qb \to \neg Pa$	→I, 5–10
12	$Raba \to (Qb \to \neg Pa$	→I, 4–11
13	$\forall y(Raya \to (Qy \to \neg Pa))$	∀I, 12
14	$\forall x \forall y(Rxyx \to (Qy \to \neg Px))$	∀I, 13

ii. $\forall x(Px \to \exists y(Ryx \land Py)), \forall x\forall y\forall z(Ryx \to (Rzy \to \neg Pz)) \vdash \neg\forall xPx$

Solution: Any valid natural deduction was accepted. Example answer:

1	$\mid\ \forall x(Px \to \exists y(Ryx \land Py))$	
2	$\mid\ \underline{\forall x\forall y\forall z(Ryx \to (Rzy \to \neg Pz))}$	
3	$\mid\ \mid\ \underline{\forall xPx}$	
4	$\mid\ \mid\ Pa$	∀E, 3
5	$\mid\ \mid\ Pa \to \exists y(Rya \land Py)$	∀E, 1
6	$\mid\ \mid\ \exists y(Rya \land Py)$	→E, 5, 4
7	$\mid\ \mid\ Pb$	∀E, 3
8	$\mid\ \mid\ Pb \to \exists y(Ryb \land Pb)$	∀E, 1
9	$\mid\ \mid\ \exists y(Ryb \land Pb)$	→E, 8, 7
10	$\mid\ \mid\ \mid\ \underline{Rba \land Pb}$	
11	$\mid\ \mid\ \mid\ \mid\ \underline{Rcb \land Pc}$	
12	$\mid\ \mid\ \mid\ \mid\ \forall y\forall z(Rya \to (Rzy \to \neg Pz))$	∀E, 2
13	$\mid\ \mid\ \mid\ \mid\ \forall z(Rba \to (Rzb \to \neg Pz))$	∀E, 12
14	$\mid\ \mid\ \mid\ \mid\ Rba \to (Rcb \to \neg Pc)$	∀E, 13
15	$\mid\ \mid\ \mid\ \mid\ Rba$	∧E, 10
16	$\mid\ \mid\ \mid\ \mid\ Rcb \to \neg Pc$	→E, 14, 15
17	$\mid\ \mid\ \mid\ \mid\ Rcb$	∧E, 11
18	$\mid\ \mid\ \mid\ \mid\ \neg Pc$	→E, 16, 17
19	$\mid\ \mid\ \mid\ \mid\ Pc$	∧E, 11
20	$\mid\ \mid\ \mid\ \mid\ \bot$	¬E, 19, 18
21	$\mid\ \mid\ \mid\ \bot$	∃E, 9, 11–20
22	$\mid\ \mid\ \bot$	∃E, 6, 10–21
23	$\mid\ \neg\forall xPx$	¬I, 3–22

(b) In the rule ∀I,

$$\begin{array}{r|ll} m & \mathbf{A}(\ldots\mathbf{c}\ldots\mathbf{c}\ldots) & \\ & \forall\mathbf{x}\mathbf{A}(\ldots\mathbf{x}\ldots\mathbf{x}\ldots) & \forall\text{I}, m \end{array}$$

why do we insist that the name **c** must not occur in any undischarged assumption? Provide an example of an invalid argument that would be provable without this constraint.

Solution: Because **c** must be a completely arbitrary name: in order to infer $\forall\mathbf{x}\mathbf{A}(\ldots\mathbf{x}\ldots\mathbf{x}\ldots)$, then it must be the case that for any other name **d**, we can prove $\mathbf{A}(\ldots\mathbf{d}\ldots\mathbf{d}\ldots)$ using the same assumptions. If **c** occurs in some undischarged assumption, then this will not typically be the case.

Without this constraint, we could prove that $Fa \vdash \forall xFx$:

$$\begin{array}{r|l} 1 & Fa \\ \hline 2 & \forall xFx \end{array}$$

(c) In the rule ∃E,

$$\begin{array}{r|lll} m & \exists\mathbf{x}\mathbf{A}(\ldots\mathbf{x}\ldots\mathbf{x}\ldots) & & \\ i & \quad\vert\ \mathbf{A}(\ldots\mathbf{c}\ldots\mathbf{c}\ldots) & & \\ j & \quad\vert\ \mathbf{B} & & \\ & \mathbf{B} & & \exists\text{E}, m, i\text{–}j \end{array}$$

why do we insist that **c** must neither occur in $\exists\mathbf{x}\mathbf{A}(\ldots\mathbf{x}\ldots\mathbf{x}\ldots)$, nor in **B**? For each constraint, provide an example of an invalid argument that would be provable without that constraint.

Solution: **c** must not occur in $\exists\mathbf{x}\mathbf{A}(\ldots\mathbf{x}\ldots\mathbf{x}\ldots)$ because it must be a name about which we have not already supposed or inferred anything: $\mathbf{A}(\ldots\mathbf{c}\ldots\mathbf{c}\ldots)$ merely tells us that **A** is true of something, not that it is true of any particular individual about which we already know something. Without this constraint, we could prove $\exists xRxa \vdash \exists Rxx$:

$$\begin{array}{r|l} 1 & Rxa \\ \hline 2 & \quad\vert\ Raa \\ 3 & \quad\vert\ \exists xRxx \\ 4 & \exists xRxx \end{array}$$

c must not occur in **B** since we must have been able to infer it from $\mathbf{A}(\ldots\mathbf{d}\ldots\mathbf{d}\ldots)$ for any other name **d**: that is, we must not be able to rely on the fact that the assumption $\mathbf{A}(\ldots\mathbf{c}\ldots\mathbf{c}\ldots)$ uses **c** in particular, rather than another arbitrarily chosen name. Without this constraint, we could prove that $\exists xFx \vdash Fa$:

$$\begin{array}{r|l} 1 & \exists xFx \\ \hline 2 & \quad\vert\ Fa \\ 3 & \quad\vert\ Fa \\ 4 & Fa \end{array}$$

3. For each of the following relations, say
   i. whether it is reflexive,
   ii. whether it is symmetric, and
   iii. whether it is transitive

   on the domain of all people alive today. (Assume that some people are wise and some are foolish, that some people are neither wise nor foolish, and that no-one is both wise and foolish; that some people live in Saffron Walden and others do not; and that no one is his own brother.)

   (a) x is wise unless y is foolish
   (b) x and y are married
   (c) x and y are brothers
   (d) x is at least 2 years older than y
   (e) x is at least 200 years older than y
   (f) x is Boris Johnson iff y is Keir Starmer
   (g) Either x has the same first name as y or x has the same surname as y
   (h) Exactly one individual in the set $\{x, y\}$ lives in Saffron Walden

Solution:

	i. Reflexive?	ii. Symmetric?	iii. Transitive?
(a)	No	No	Yes
(b)	No	Yes	No
(c)	No	Yes	No
(d)	No	No	Yes
(e)	No	Yes	Yes
(f)	No	No	No
(g)	Yes	Yes	No
(h)	No	Yes	No

A mark was deducted (from 20) for each error made. So, a candidate who got each property correct for each relation would receive 20 marks, while a candidate who got every property wrong bar four would receive 0 marks.

4. You draw three cards without replacement from a pack of ten cards numbered 1–10. What is the probability that:
   (a) The first is higher than the second?
   (b) All three cards add up to 8?
   (c) The second is even, given that the first and the third are even?
   (d) The third is higher than the second, given that the second is higher than the first?

Solution:

(a) $\frac{1}{2}$. There are $10 \times 9 = 90$ ways to draw two cards without replacement. If the first card is 10, there are nine ways to select a lower card; if it is 9, there are eight ways; etc. So the number of ways to pick out two cards such that the first is higher than the second is $9+8+7+6+5+4+3+2+1 = 45$. So, the probability is $45/90 = \frac{1}{2}$.

More elegantly: for every pair of cards such that the first is higher than the second, there is a corresponding pair such that the second is higher than the first. So the probability that the first is higher than the second must be equal to the probability that the second is higher than the first. Since we are drawing without replacement, these are the only two possibilities, so their probabilities must sum to 1; hence, they have probability $\frac{1}{2}$ each.

(b) $\frac{1}{60}$. There are $10 \times 9 \times 8 = 720$ ways to draw three cards without replacement. The only combinations that add up to 8 are the sets $\{1, 3, 4\}$ and $\{1, 2, 5\}$; since each has three members, the number of different orders in which one could draw each set is $3 \times 2 \times 1 = 6$. So there are 12 ways to draw three cards such that they add up to 8, yielding a probability of $12/720 = 1/60$.

(c) $\frac{3}{8}$. The probability of drawing all three cards as even is $\frac{5}{10} \times \frac{4}{9} \times \frac{3}{8} = \frac{1}{12}$. There are two (mutually exclusive and jointly exhaustive) ways to draw the first and third cards as even: with the second card as even, and the second card as odd. The probability of the latter is $\frac{5}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{5}{36}$. So the probability of drawing the first and third cards as even is $\frac{8}{36}$; and hence, the probability that the second card is even, given that the first and the third are even, is $\frac{1}{12}/\frac{8}{36} = \frac{3}{8}$.

More elegantly: suppose that the first card had been drawn as even, and that you knew that the third card to be drawn would be 2. Then the probability of drawing an even card second would be $\frac{3}{8}$, since there are only 8 ‘available’ cards (that haven’t already been drawn first and are not about to be drawn third), of which 3 are even. The same would be true if one knew that the third card would be a 4, a 6, an 8 or a 10: hence, in any such case, the probability of the second card being even would be $\frac{3}{8}$. Since these possibilities exhaust all the cases in which the first and third cards are even, it follows that the probability of the second card being even given that the first and third are even is $\frac{3}{8}$.

(d) $\frac{1}{3}$. The probability that the second card is higher than the first is $\frac{1}{2}$ (as per part (a)); so, we need to compute the probability that the third card is higher than the second and the second is higher than the first.

As per part (b), there are 720 ways to draw three cards. If the first card drawn is a 1, then—by the same reasoning as in part (a)—the number of ways to draw the second and third cards such that the second is lower than the third is $8 + 7 + \cdots + 1 = 36$. If the first card drawn is a 2, then (by the same reasoning again) the number of ways is $7 + 6 + \cdots + 1 = 28$. If the first card is a 3, we get 21 ways; if it is a 4, we get 15 ways; if it is a 5, we get 10 ways; if it is a 6, we get 6 ways; if it is a 7, we get 3 ways; and if it is an 8, we get 1 way. So the total number of ways for the first card to be lower than the second, which is in turn lower than the third, is $36 + 28 + 21 + 15 + 10 + 6 + 3 + 1 = 120$. Hence, the probability of this event is $\frac{120}{720} = \frac{1}{6}$.

More elegantly, suppose that any particular set of three distinct cards were drawn: say, the set $\{2, 5, 7\}$. There are six possible orders in which these cards could have been drawn, one and only one of which involves them being drawn such that the second is higher than the first and the third is higher than the second (namely, the order $(2, 5, 7)$). So conditional on those cards being drawn, the probability of the third being higher than the second and the second being higher than the first is $\frac{1}{6}$. But the same is true for any other set of three cards that we might want to consider. So overall, the probability of the third being higher than the second and the second being higher than the first is $\frac{1}{6}$.

Either way, we can now compute the answer: we find that the probability that the third card is higher than the second given that the second is higher than the first is $\frac{1}{6}/\frac{1}{2} = \frac{1}{3}$.