SETS RELATIONS & PROBABILITY LECTURE 2

1. A convenient bit of notation is to write $\{x \mid Fx\}$ for the set of F's i.e. the set of xs such that Fx (some writers have '$\{x: Fx\}$'). The next definition then says that if X is a set then there exists a set $\wp(X)$, called the 'Power Set' of X, whose members are just the subsets of X. In short:

   $$\wp(X) = \{Y \mid Y \subseteq X\} = \{Y \mid \forall z\ (z \in Y \rightarrow z \in X\}$$

   (The right hand side is just a definitional expansion of '$\subseteq$'.) Note that we must not confuse the subsets of a set with its members: the members of $\wp(X)$ are typically not members of X itself, though they may be. Now it can be shown (exercise for the reader) that if a finite set X has n members then its power set $\wp(X)$ has 2^n members. It can therefore be shown that $\wp(X)$ has strictly more members than X; and Cantor showed that this remains true, in a sense, for infinite sets (his famous 'diagonal' proof), thus inaugurating transfinite arithmetic.

2. The next definition is of the ordered pair. The ordered pair of a and b, written (a, b), is the set $\{a, \{a, b\}\}$ (it can also be defined as $\{\{a\}, \{a, b\}\}$). Note that the ordered pair (a, b) is not the same thing as the set $\{a, b\}$. This follows from the fact that $\{a, b\} = \{b, a\}$ (which is true by extensionality) but $(a, b) \neq (b, a)$ (by extensionality and the fact that a (or $\{a\}$) is a member of (a, b) but not of (b, a)). The key thing about this definition is that it is sensitive to the order of its components.

3. If A and B are sets then there is a set $A \times B$, called the *Cartesian Product* of A and B, whose elements are just the ordered pairs whose first member belongs to A and whose second member belongs to B:

   $$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

   We write A^2 for $A \times A$ e.g. as it might be co-ordinates on a map.

4. Two final bits of notation: there is (by extensionality) just one set $\{\ \}$ that has no elements at all: this is written $\varnothing$ and is called the *empty set*. And for any sets A and B, there is a set of objects that belong to A but not to B. This is the *complement* of B in A, written:

   $$A - B = \{x \mid x \in A \wedge \neg x \in B\}$$

   Note that writers sometimes use $x \notin A$ for $\neg x \in A$.

5. We can use these definitions to write down set-theoretic expressions for many combinations of sets. If e.g. we use C, R and W for the sets of students at Caius, Robinson and Wolfson, and M and F for the set of male persons and the set of female persons then e.g. $F \cap R$ is the set of female persons at Robinson, $C \cup W$ is the set of all individuals that are either at Caius or at Wolfson (or both) and $M \times F$ is the set of all possible male-female marriages.

6. One application of set theory is the treatment of properties and relations. Properties are ordinarily conceived of as *intensional*: you can have two properties possessed by exactly the same objects. Consider e.g. the properties of being renate (having a kidney) and being cordate (having a heart). Any animal that possesses either property possesses both. But they are different properties, for it is at least possible that something should have one but not the other.

7. But we may care only about what is called the *extension* of a property. Two properties have the same extension iff everything that has either has both. We identify the extension of a property P with the set of objects bearing it, $\{x \mid Px\}$. If e.g. P is the property of being renate and Q that of being cordate then P and Q have the same extension, for $\{x \mid Px\} = \{x \mid Qx\}$.

8. What is the extension of a relation? It can't be the set of objects that fall under it, because it isn't objects but *pairs* of objects that fall under it. Maybe the extension of a relation R is the set of all *sets* {a, b} such that Rab? That is no good because it doesn't distinguish the (extension of the) relation *x is younger than y* from *x is older than y*. What we need is *ordered* pairs or *coordinates*.

9. We may now define the extension of a relation R as the set of ordered pairs (x, y) such that Rxy, or $\{(x, y) \mid Rxy)$. For example, if R is the relation *x is bigger than y* then the ordered pair (Russia, England) is in the extension of R; the ordered pair (England, Russia) is *not*. Note that the empty set is a relation; it is the *empty relation* and it holds between no objects at all (e.g. $x=y \wedge \neg x=y$).

10. It is worth going over some examples to illustrate the applications of the symbolism. Give examples of:

    (a) A set with exactly two members
    (b) A set with a power set that has exactly two members
    (c) A set with a power set that has exactly one member
    (d) Two sets whose Cartesian product has exactly four members

    You will also need to be able to switch between different forms of notation (and English). For instance:

    (e) If X = {Alice, Bob} and Y = {Bob, Charlie, Diana} then what is $X \cup Y$? What is $X \cap Y$?
    (f) Write out $\wp(\wp(\varnothing))$. Write out $\wp(\{a, b, c\})$.
    (g) If M is the set of males, F the set of females and C the set of children, write an expression for the set of all people.
    (h) In the notation of (g), write an expression for the set of all groups of adult males.
    (i) Alice is older than Bob who is older than Charlie. What is the extension of the relation *x is older than y*? What is the extension of the relation *x is at least as old as y*?