SETS RELATIONS AND PROBABILITY LECTURE 3

1. **Relations.** A relation is defined on a *domain*: the set of objects that we are thinking of applying the relation to. It might be e.g. the set of people (or e.g. some set of numbers, or a completely various set). All the quantifiers in what follows are understood as ranging over a pre-specified domain (this will become clearer when you see some examples).

2. The theory of relations involves certain basic ways of classifying relations: reflexive, symmetric, and transitive. We say that a relation R is *reflexive* if *everything* in the domain bears R to *itself* i.e.$\forall x$ (Rxx). E.g. the relation *x is no bigger than y* is reflexive for any domain. R is *anti-reflexive* if *nothing* in the domain bears R to itself. E.g. the relation *x is at least one stone heavier than y* is anti-reflexive. Note that some relations are *neither* reflexive *nor* anti-reflexive; if the domain is not empty then no relations are both. A special relation holds between any a and b if and only if every reflexive relation does. What is it called?

3. We say that a relation R is *symmetric* if whenever x bears R to y, y bears R to x i.e. $\forall x \forall y$ (Rxy $\leftrightarrow$ Ryx). The relation *x is married to y* is symmetric, if the domain is the set of people. A relation is *anti-symmetric* if $\forall x \forall y$ (Rxy $\rightarrow \neg$Ryx). The relation *x is the father of y* is anti-symmetric if the domain is people. The empty relation is symmetric *and* anti-symmetric.

4. We say that a relation is *transitive* if whenever x bears the relation to y and y bears it to z, then x bears it to z: $\forall x \forall y \forall z$ ((Rxy $\wedge$ Ryz) $\rightarrow$ Rxz). Thus the relation *x is the father of y* is *not* transitive, but the relation *x is an ancestor of y is* transitive. Similarly, if $\forall x \forall y \forall z$ ((Rxy $\wedge$ Ryz) $\rightarrow$ $\neg$Rxz), we call R *anti-transitive*. The empty relation is both transitive and anti-transitive. Many relations that you would have thought were transitive turn out not to be: this can be both surprising and interesting.

5. Two points deserve emphasis. First, whether a relation is, say, reflexive, depends on the domain of objects that it is being applied to. For example, on the domain of *people* the relation *x loves y* is perhaps not reflexive as there are people who do not love themselves. But if we restrict the domain to e.g. the set of those people that are attending a convention of narcissists then plainly the relation *is* now reflexive.

6. The second point is that whether a relation is reflexive or not depends only on what there actually is, not on what there might be. The relation of identity is plainly reflexive and necessarily so; but there are some relations that are in fact reflexive but might not have been. Consider the relation *x won at Austerlitz* $\leftrightarrow$ *y lost at Waterloo*. This is a reflexive relation (on the domain of people), because in fact everybody in the world has it to him- or herself. But it need not have been, and it *would* not have been if the victor of Austerlitz had not also lost at Waterloo. But it *is* reflexive. Both of these points apply equally to symmetry and transitivity.

7. The third point is that, as the previous example illustrated, *any* open sentence with an *x* and a *y* occurring free in it denotes a relation. The following is a relation *x was US President in 1864 and y was US President in 2008*. That relation holds between Abraham Lincoln and George W. Bush only (in that order) but it is none the less a *relation*. In fact *any* set of ordered pairs counts as a relation.

8. You should be able to classify relations according to whether they are reflexive, symmetric and/or transitive given a specific domain. For instance:

   (a) *x is older than y* on the domain of currently living people
   (b) *x and y have the same Christian name* on the same domain
   (c) *x and y have the same surname or Christian name* on the same domain
   (d) *x and y are brothers* on the same domain
   (e) *x is male and y is female* on the same domain
   (f) *x wrote Waverley → y wrote Persuasion* on some domain of which the authors of those novels are both elements

9. You should also be able to give examples of relations that have specific properties. When you do this, you should always be careful to specify the domain. For instance:

   (a) Give an example of a relation that is reflexive, symmetric and transitive
   (b) Give an example of a relation that is symmetric and transitive but not reflexive
   (c) Give an example of a relation that is reflexive and symmetric but not transitive
   (d) Give an example of a relation that is not reflexive, symmetric or transitive

10. A relation that is reflexive, symmetric and transitive is called an *Equivalence Relation*. These relations are very important in many areas of science and philosophy. Examples of equivalence relations include *x is (exactly) as tall as y*, *x is (exactly) as old as y*, *x was born in the same town as y*, and *x*=*y*. It is easy to verify that these relations are reflexive, symmetric and transitive. The following relations are *not* equivalence relations: *x is married to y* (not reflexive), *x is a brother of y* (not symmetric), *x was born in or died in the same town as y* (not transitive).