SETS, RELATIONS AND PROBABILITY LECTURE 4

1. If R is an equivalence relation on a certain domain then there is a special *partition* on that domain. The notion of partition can informally be described like this: it is a way of dividing up the elements of a domain into sectors in such a way that every element belongs to exactly one sector. This means that the sectors are non-overlapping and include everything, or as we say: the division is exclusive (nothing belongs to more than one sector) and exhaustive (everything belongs to at least one sector). An example of a partition is the way the Allies divided up Berlin after WWII.

2. Slightly more formally, we can say that a partition is a set of such sectors, each sector itself being a set: that is: if X is a set then a partition on X is a set P of subsets of X such that:

   $\forall x\ (x \in X \rightarrow \exists p\ (p \in P \wedge x \in p \wedge \forall q\ ((x \in q \wedge q \in P) \rightarrow q = p)))$

3. These sectors are called R's *equivalence classes*. E.g. if R is the equivalence relation *x was born in the same country as y* then the equivalence classes are just these: the set of people born in the UK, the set of people born in France, … etc. If R is the equivalence relation on points on the surface of the Earth *x is the same height as y* then the equivalence classes are marked on a map as contour lines (that is why contour lines never meet). And Russell and Frege defined cardinal numbers as equivalence classes of *sets* on the equivalence relation *X and Y are equinumerous*. Question: how big are the equivalence classes of the identity relation?

4. I shall introduce a little more notation. If R is a relation then we say that the *converse* of R, sometimes written R', is that relation that holds between *x* and *y* if and only if R holds between *y* and *x*. I.e. R' is the converse of R iff $\forall x \forall y\ (Rxy \leftrightarrow R'yx)$. Until recently in the UK, *x is the husband of y* was the converse of *x is the wife of y*; it remains true in certain Islamic countries that *x is the husband of y* is the converse of *x is <u>a</u> wife of y*. Similarly, *x is smaller than y* is the converse of *x is bigger than y*; and *x is above y* is the converse of *x is beneath y*. What do we call relations that are their own converses?

5. If R is a relation then the *ancestral* of R, sometimes written R*, is that relation that holds between *x* and *y* if and only if for some *z*, *w*, *v*, *u* ... we have R*xz* & R*zw* & … & R*vu* & R*uy*. For instance, if R is *x is a parent of y* then R* is *x is an ancestor of y*. What do we call relations that are their own ancestrals?

6. A straightforward application of the second point is to Locke's theory of personal identity. We know from Reid's objection that the memory-relation is not transitive. Plainly however identity is. Clearly then the relation Rxy = *x remembers being y (or vice versa)* can't be the same relation as *x and y are the same person*. But no such difficulty arises for R*.

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7. We can now move on to **probability**. There are two important things that you will eventually want to learn about: the probability calculus, and the interpretation of that calculus, and this course is mostly about the first. The nature of this distinction will become clearer when you have tried to use the calculus for yourself.

8. What are the things to which we assign probability? If you say that the probability that the Conservatives win the next election is 40%, are you saying it of an event, a proposition, a sentence, or a set? Here we shall say that it is sets of outcomes. In some ways it is natural to regard a Conservative victory as a set of outcomes. For it includes the outcome that they win by one seat, the outcome that they win by 100 seats etc., and these are different. It isn't fixed a priori *how* finely one should divide possibilities into outcomes but is usually settled by the context of the investigation, as we shall see.

9. We begin by defining the universe of possibilities that we wish to consider. This is a set V of **outcomes**, and V is called the **reference set** or **sample space**. For example, if we are throwing a die, there are 6 possible outcomes, and V = {1, 2, 3, 4, 5, 6}.

10. But there are many more than 6 possible *sets* of outcomes or **events**. We might want to know the probability of an even number, for example, or that of: a 1 or a 6. We may write these situations as {2, 4, 6} and {1, 6} respectively. The **field** F contains all such sets: F = $\wp$(V). Note in particular that it contains the certain event V = {1, 2, 3, 4, 5, 6} and the impossible event $\varnothing$. Note also that if $X \in F$ and $Y \in F$ then $X \cap Y \in F$ and X U Y $\in$ F. For any $X \in F$ I'll write X* for V - X i.e. the set of all outcomes other than those that are members of X.

11. Now *probability*, or better a *probability function*, is a function Pr that assigns numbers to members of F and which obeys the following axioms for any X, Y $\in$ F:

    (i) Pr (V) = 1
    (ii) Pr (X) ≥ 0
    (iii) If $X \cap Y = \varnothing$ then Pr $(X \cup Y)$ = Pr (X) + Pr (Y)

    Axioms (i)-(iii) are known as the **Kolmogorov axioms** (in fact (iii) is slightly simplified in ways that are unimportant for present purposes). Given these axioms we can show that Pr (X) + Pr (X*) = 1. (You won't need to know how to do this for the exam.)

12. Probabilities may now be assigned to events. First we assign probabilities to the outcomes themselves, or rather to their singletons; logic does not tell us what these are. You might think they could be settled a priori on the basis that if there is no reason to choose between two options then they are equally likely. Unfortunately that principle is inconsistent.