SETS, RELATIONS AND PROBABILITY LECTURE 7

1. Now we turn to conditional probabilities in dependent trials. For instance, we can try calculating the probability that if two cards are drawn from a pack without replacement, both are aces given that one of them is an ace. Let C say that they are both aces and let X say that one of them is an ace. Then we already know that:

   $\Pr(C) = 12 / (51 \times 52) = 1 / 221$
   $\Pr(X) = 33 / 221$

We also know that $C \cap X = C$ and so:

   $\Pr(C \mid X) = \Pr(C \cap X) / \Pr(X) = \Pr(C) / \Pr(X) = 1/33$

2. Now try working out the probability that they are both aces given that one of them is *the ace of spades*. Let Y say that one of them is the ace of spades and consider the following sets:

   Y_1: The first is the ace of spades (and the second is not): 51 outcomes
   Y_2: The second is the ace of spades (and the first is not): 51 outcomes
   $C \cap Y_1$: First is ace of spades, second is another ace: 3 outcomes
   $C \cap Y_2$: Second is ace of spades, first is another ace: 3 outcomes

   Hence:

   $\Pr(Y) = \Pr(Y_1) + \Pr(Y_2) = 102 / (51 \times 52)$
   $\Pr(C \cap Y) = \Pr(C \cap Y_1) + \Pr(C \cap Y_2) = 6 / (51 \times 52)$; hence
   $\Pr(C \mid Y) = 6 / 102 = 1 / 17$.

   Here too the conditional probability is highly sensitive to what is being conditionalized on. Can you explain why the answer to 2 is nearly twice the answer to 1?

3. So far we have looked mainly at mechanical devices for calculating probabilities of events of interest on the basis of given probabilities for other events. We have considered the idea that probabilities might be interpreted as frequencies; but another interpretation of especial interest is the following: the probability that you assign to an event E is a measure of your *degree of confidence* that E occurs. The analysis that I have in mind is due to Frank Ramsey ('Truth and Probability' in his *Philosophical Papers* ed. D. H. Mellor).

4. First we need to know something about odds given by bookies. If Ladbrokes offers, say, 1:1 on Labour winning the next election, that means that if you bet £50 on it and it happens you will get your £50 back plus another £50. If it doesn't happen then you lose your stake. Similarly, if they offer odds of 8:5 then of your £50 bet wins you will end up with £130 (£50 stake + £80 winnings). More generally, if they offer odds of A: B then if you win you will end up with £(A + B)/B for every pound that you stake. Now obviously some odds are better than others:

for instance, odds of 10:1 are better (from your point of view) than odds of 5:1. Better odds are ones where (A+B)/B is higher.

5. The degree of confidence, or **credence**, that you have in an event E is determined by the *worst* odds on E that you would accept, at least when *small* stakes are involved. For instance, if the worst odds on E that you would accept are 2:1 then that means that you have a degree of confidence of 1/3 in E. More generally, if the worst odds that you are prepared to accept is A:B then your confidence in E is B/(A+B). This makes intuitive sense: the more likely you think that something is to happen, the worse the odds at which you would bet on it. Thus if you are certain that something will happen then you will accept *any* bet on it at odds A:B (however small A is, as long as it is not zero); if you are certain that it will *not* happen you will accept *no* bet on it at odds A:B (however small B is, as long as it is not zero).

6. If the probability symbol Pr is interpreted as a measure of your fair betting odds, then the Kolmogorov axioms, and hence the theorems, of the probability theory have a kind of rational compulsion.

7. To see this we need to grasp the notion of a **Dutch Book**. We say that you are open to a Dutch Book if there is a set of betting odds on events that are arranged in such a way that you are both willing to take them and bound to lose money. Bookies as well as bettors can be open to Dutch books. Thus suppose for instance that Ladbroke's offers 3:1 on Red Rum, 3:1 on Shergar and 3:1 on Logic Boy, and suppose these are the only horses in the race. What you should do is put 1/3 of your capital on *each* of them: that way you are bound to win.

8. The sense in which the axioms of probability are rationally compulsory depends on the point about Dutch Books. If we interpret probabilities subjectively then anyone who violates the axiom is vulnerable to a Dutch Book: that is to say, an intelligent punter could *always* make money out of them. For example, suppose that my confidence in E and ¬E respectively is Pr (E) = 0.75 and Pr (¬E) = 0.75. Then clearly I have violated the theorem that Pr (X) + Pr (¬X)=1.

9. Now why does this lead me into trouble? Well, it means that I will accept odds of 1:3 on E (say, rain tomorrow) and odds of 1:3 on ¬E. A cunning bookie will then simply offer me the chance to bet £3 on E and £3 on ¬E at those odds. I will take both bets. But then if E happens the bookie will lose £1 on the first bet and take £3 on the second; and if E does *not* happen the bookie will take £3 on the first bet and lose £1 on the second. Either way he has done me out of £2. The kind of economic irrationality that leaves me open to such exploitation is rather like what we saw in the case of intransitive preference. It is the avoidance of this kind of irrationality that is supposed to justify our accepting the laws of probability when these are interpreted as degrees of belief.