Inaccessible truths and infinite coincidences

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Abstract

It is one of the salient features of an intuitionistic philosophy of mathematics that it denies the possibility that there may be infinite coincidences in mathematics: if a sentence is true, it can only be because there is a finitely expressible reason for it. The same view was expressed from a different standpoint by Hilbert in 1924 when he asserted that in mathematics there is *ignorabimus*. It has been common since Gödel's incompleteness theorems to regard this view as no longer open to the platonist. And yet it has been reaffirmed since then by Gödel himself in regard to unsolved problems in set theory such as that of settling the continuum hypothesis. It is therefore a live question whether there is a coherent position which denies this. We should distinguish three possibilities such a position might be intended to allow: first, sentences true accidentally, by infinite coincidence; second, truths which are in principle inaccessible to us and which we cannot even grasp directly via our intuitions about the concepts involved; third, truths for which there is no finitely expressible reason. The first possibility implies the second, and the second implies the third.

It is one of the salient features of an intuitionistic philosophy of mathematics that it denies the possibility that there may be infinite coincidences in mathematics. If a sentence is true, then according to the intuitionist account that can only be because there is a reason why it is true, a reason which we — as creative mathematicians — must be capable in principle of grasping.

Until 1931 this view — that every true mathematical sentence is in principle capable of being known to be true by us, or at any rate by our ideal counterparts — would also have found support from mathematicians not otherwise sympathetic to constructivism. Frege certainly believed it. 'In arithmetic,' he wrote in *Grundlagen* ([3], §105), 'we are not concerned with objects which we come to know as something alien from without through the medium of the senses, but with objects given directly to our reason and, as its nearest kin, utterly transparent to it.' And Hilbert in his 1924 lecture "Über das Unendliche" ([1], p.200) remarked that 'one of the principal attractions of tackling a mathematical problem is that we always hear this cry within us: There is the problem, find the answer; you can find it just by thinking, for there is no *ignorabimus* in mathematics.'

But this, of course, was before Gödel had proved his incompleteness theorems. Since 1931 it seems to have been common for those going under the name of 'moderate realists' ('moderate' presumably in distinction to immoderate realists like Gödel) to hold that Frege and Hilbert were wrong, and to believe instead that there are, or at any rate may be, true mathematical sentences which it is beyond the capabilities of any finite intelligence to recognise as true.

It is unlikely, however, that Gödel himself believed this. His published writings on the philosophy of mathematics are of course very limited, totalling only a few pages, and none of them address the question in quite the form in which I have stated it. Nevertheless, I want to discuss here what reasons a platonist such as Gödel — although in advance of the publication of his *Nachlass* papers in Volume III of the *Collected Works* it would be foolhardy of me to say Gödel himself — might have for agreeing with the intuitionist on this matter. What will emerge, I think, is that the similarities between the views of the Dummettian intuitionist and the Gödelian platonist are greater, and the so-called moderate realist is less moderate, than is generally imagined.

We are interested, then, in the possibility that there may be true sentences for which there is no finitely expressible reason. Another possibility, which implies this one, is that there may be true sentences which we cannot know. And yet a third, which implies the second, is that there may be sentences true accidentally, despite there being nothing in our current grasp of the concepts involved in their expression from which it follows that they are true. I intend to examine each of these possibilities in turn, starting with the last.

The image of mathematical sentences being true by accident is an arresting one. It is plainly repugnant to anyone who believes in a fundamentally ordered universe. That, however, is not in itself a sufficient reason to reject it.

In the case of most of the sorts of sentence which interest us we can reject it for a much more clear-cut reason, namely that God simply does not, under the platonist interpretation of the quantifiers on the natural numbers, have the freedom to decide their truth or falsity, whether by dice-throwing or the exercise of God's whim or anything else. A sentence of Goldbach type (i.e. of the form $\forall x \phi(x)$ where $\phi(x)$ is a mechanically decidable arithmetical predicate), for example, is decided for each natural number by a decision procedure which is out of God's hands. The truth of the generalization, even if it is unknowable by us, cannot be a matter on which God could arbitrate. Similar considerations apply to each more complex sort of arithmetical sentence: at each stage the truth-conditions for the sentences in question, realistically conceived, determine their truth or falsity, leaving no scope for the hand of God.

The same argument does not work in the case of the continuum hypothesis. In this case there is more appeal, superficially at any rate, in the notion that there could be a fact of the matter completely unconstrained by the grasp that we mere mortals have of the concepts involved. This view could be intelligible, however, only to what I shall call a metaphorical platonist, that is to say someone who treats the analogy between the existence of physical objects and the existence of mathematical ones seriously as a literal account of the way things are.

Now I have already remarked that a sentence true only by God's decree is in principle unknowable. This is because nothing short of divine revelation could account for such knowledge. But if we cannot, save by divine intervention, know some of the truths of arithmetic, how are we to explain our knowledge of the others? As soon as we accept the image of God constructing set theory, and exercising free choice in how He constructed it, we must allow the possibility that He chose not to construct it at all. So even the knowledge we do have of set-theoretic truths requires for its explanation an appeal to the same extravagant mental powers of contact with the mind of the Almighty that would be needed to account for knowledge of the accidental truths.

It is plain that this is absurd. The platonist metaphor is not to be taken so seriously. However, a literal belief in this metaphor, and in the mysterious mental powers necessary to make it plausible, has been widely attributed to Gödel. After all, in the second edition of "What is Cantor's continuum problem?" ([4], p.268) he famously wrote: 'Despite their remoteness from sense experience we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true.' This is, I think, the most misunderstood sentence Gödel ever wrote.

The explanation of the thing 'like perception' which on Gödel's view we have of the objects of set theory is provided in the following paragraph, where he says: 'Mathematical intuition need not be conceived of as a faculty giving an *immediate* knowledge of the objects concerned. Rather it seems that, as in the case of physical experience, we *form* our ideas also of those objects on the basis of something else which *is* immediately given. Only this something else here is *not*, or not primarily, the sensations. That something besides the sensations actually is immediately given follows (independently of mathematics) from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, e.g., the idea of object itself, whereas, on the other hand, by our thinking we cannot create any qualitatively new elements but only reproduce and combine those that are given.'

So for Gödel what is involved in the exercise of mathematical intuition is just what is needed to close the gap between Kant's notion of analyticity, according to which analytic truths are trivial, and the perceived fruitfulness of mathematical reasoning. Dummett's account of what is required to close this gap is based on the discernment of pattern. 'The pattern,' he wrote in *The logical basis of metaphysics* ([2], p.198), 'is not, in general, *imposed*: it is *there* to be discerned; we can be fully aware of that without apprehending the pattern.' What Dummett has in mind here is the distinction between recognising a proof as correct and understanding the proof. What is involved in this is not 'merely to grasp the thought expressed by each line of the proof; in addition, one must perceive patterns common to those thoughts and others, patterns which are not given with the thoughts as a condition for grasping them but which require a further insight to apprehend.'

Dummett is here expounding Frege's thought, but it could just as well be Gödel's. On Gödel's account in his *Dialectica* paper, for example, what enables mathematics to go beyond finitism is our ability to grasp abstract concepts, by which he means ([4], pp.272–3) 'concepts which are essentially of the second or higher level, i.e., which do not have as their content properties or relations of *concrete objects* (such as combinations of symbols), but rather of *thought structures* or *thought contents* (e.g. proofs, meaningful propositions, and so on), where in the proofs of propositions about these mental objects insights are needed which are not derived from a reflection upon the combinatorial (space-time) properties of the symbols representing them, but rather from a reflection upon the *meanings* involved.'

The similarity between Gödel's thought here and Dummett's is, I hope, clear. In particular, although the first quotation from Dummett, where he refers to the discernment of patterns might out of context suggest that Dummett has in mind only insights 'derived from a reflection upon the combinatorial (space-time) properties of the symbols', the other quotation makes clear that Dummett intends to include the patterns which we see when we understand the symbols occurring in a proof (say) as part of the logical structure of the proof. In other words, to perceive the patterns Dummett is here referring to we must not merely see the symbols before us as exemplifying thoughts, but also be capable of reflecting upon our own grasp of those thoughts. Nothing less, on Dummett's view just as on Gödel's, can account for our experience of deductive progress.

We come now to the question of whether the platonist should, like the intuitionist, affirm that all true sentences are knowable in principle. Let us note straightaway that if he does, he must refuse to assent to the possibility that the human mind functions like a machine, since if it did the criteria which determine whether a sentence is in principle knowable would be formalizable and therefore susceptible to Gödelian diagonalization to obtain a true unknowable sentence. (The intuitionist faces essentially the same consequence, although he expresses it a little differently.)

It is interesting in this connexion to note Gödel's reason for rejecting an argument of Turing's intended to establish that mental procedures cannot go beyond mechanical ones. He argued ([4], p.306) that 'mind, in its use, is not static, but constantly developing, i.e., that we understand abstract terms more and more precisely as we go on using them, and that more and more abstract terms enter the sphere of our understanding.' He gives as an example of this the process of forming stronger and stronger axioms of infinity in set theory. In other words, what in Gödel's view may mark out the operation of the human mind as essentially different from that of a machine is its ability not merely to perceive physical properties of strings of symbols but to reflect on its own grasp of the thoughts which we take these strings of symbols to express, and as a result to be guided to new abstractions. One is drawn irresistibly to the thought that part of what Gödel is here referring to is what Dummett has described as our ability to grasp indefinitely extensible concepts.

The problem of mechanism about the mind is closely related to the question whether the knowable sentences are those for which there are finitely expressible reasons. Gödel would presumably have argued that the stronger and stronger axioms of infinity to which we are willing to assent are examples of sentences for which there is no finitely expressible reason. We assent to them because we recognise them to be axioms of infinity and therefore true according to the iterative conception of set theory, but we cannot explain precisely why. For suppose that we could give a general account of what it is to be an axiom of infinity of the sort which we ought to accept as true. If that general account could be formalized, then we would add it to the formal system for set theory: there would then be further axioms of infinity not coming within the compass of the notion we had formalized, and the question of acceptance would arise for them. If, on the other hand, the general account could not be formalized, then recognising that our putative axiom is indeed an instance of the general notion would be a non-mechanical matter, and we could presumably give no finitely expressible reason for *that*.

It is possible to imagine there being in arithmetic as well examples of sentences we can know without there being a finite reason why they are true. There might be a sentence $\forall x\phi(x)$ of Goldbach type such that the mechanical verifications that $\phi(0), \phi(1), \phi(2), \ldots$ are all true display a regularity, a pattern which we can recognise but which we cannot describe using the conceptual apparatus currently available to us. In such circumstances we would, in the process of seeing the pattern, come to form a new concept which would allow us to recognise the generalization as true. I think this is the sort of situation Gödel had in mind when he referred ([4], p.269) to 'the fact that continued appeals to mathematical intuition are necessary not only for obtaining unambiguous answers to the questions of transfinite set theory, but also for the solution of the problems of finitary number theory (of the type of Goldbach's conjecture), where the meaningfulness and unambiguity of the concepts entering into them can hardly be doubted.'

Let us return finally to the central question of whether every true arithmetical sentence is knowable. The possibility, among the three I outlined before, which remains to be addressed, in the case of arithmetic, is of a sentence $\forall x\phi(x)$ as in the example we considered a moment ago, except this time there is no regularity: there is no pattern in the calculations leading to the conclusions that $\phi(0), \phi(1), \phi(2), \ldots$ are each true.

In such a case one might want to say that there is an infinite reason why $\forall x \phi(x)$ is true, namely the infinite concatenation of the proofs of $\phi(0), \phi(1), \phi(2), \ldots$, but of course that is of no help to *us* in *our* attempt to grasp the truth of the generalization, however helpful we may imagine it to be to God in His.

Nor are the considerations which led us to reject metaphorical platonism relevant here. There is no suggestion in the case we are imagining now of the truth of the Goldbach sentence outstripping the grasp we have of the concepts involved in its statement. The generalization is true because each of its instances is true, and each instance is true because a mechanical calculation shows it to be true. The intuitionist, of course, will say that the only way we can understand the generalization is as a claim that there exists a schema which for each n provides us with a proof of $\phi(n)$. But that is of no help to the platonist, who understands the universal quantifier over the natural numbers differently.

For some, there is no need of an argument. I have spoken to philosophers who regard it as the plainest of trivialities that if the truth of an arithmetical sentence answers to the grasp we have of the concepts it involves, then we must be capable in principle of recognising that fact. For myself, I cannot go as far as this. If we are to construct an argument for it, I think we should start by noting how difficult it is even to *state* the possibility we have in mind. I spoke just now about a sentence whose truth 'answers to the grasp we have of the concepts it involves'. That was intentionally vague. It would be more natural to say 'is a consequence of', but that is to give the game away, since presumably it is part of what we understand a consequence to be that we are capable of grasping it. Earlier I used the word 'because', and the same argument applies to that way of talking too.

So let us try again. In the case we have in mind the sentences $\phi(0), \phi(1), \phi(2), \ldots$ are all instances of one decidable predicate $\phi(x)$. There is therefore a schematic calculation involving the variable x which, whenever a particular natural number n is substituted for x and the resulting explicit calculation is undertaken, results in the output 'True'. We might be tempted to describe that fact — that each instance of the schematic calculation results in the output 'True' — as being in itself a pattern. And yet, in the case in question, we have to imagine that it is a pattern which we are in principle incapable of recognising. But what could we mean by this — a pattern which we cannot recognise? Surely it is part of what we mean by a pattern that it should be capable of recognition. If, then, we were wrong to describe the fact that all instances of the schema come out true as a pattern, what should we describe it as? A regularity which is not a pattern? If we say that it is an accident of nature, we are drawn back into a metaphor whose inappropriateness we have already remarked on.

Each time, then, it seems that we have failed to formulate coherently the possibility we are being asked to consider, namely that of arithmetical sentences whose truth does not outstrip our understanding of what they assert but is nevertheless opaque to us. Whether failure to formulate a possibility coherently is proof that the possibility does not exist is, of course, somewhat dubious, but to those for whom the conclusion is not just plain obvious, it may be the nearest to an argument that can be offered.

References

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