

Classical arithmetic is part of intuitionistic arithmetic

Michael Potter

One of Michael Dummett's most striking contributions to the philosophy of mathematics is an argument to show that the correct logic to apply in mathematical reasoning is not classical but intuitionistic. In this article I wish to cast doubt on Dummett's conclusion by outlining an alternative, motivated by consideration of a well-known result of Kurt Gödel, to the standard view of the relationship between classical and intuitionistic arithmetic. I shall suggest that it is hard to find a perspective from which to arbitrate between the competing views. Let me start, then, by stating the standard view of the relationship, with which the account I shall be canvassing is to be contrasted.

1 The standard view

Although some of what I shall be saying can be applied to areas of mathematics other than the arithmetic of the natural numbers, much of it depends on a feature of arithmetic that is not readily generalizable, namely that its atomic sentences are decidable. It will therefore simplify matters greatly to restrict our attention solely to the arithmetical case. The relationship between classical and intuitionistic mathematics in general is complicated by the fact that the intuitionistic version results from the application of two competing principles, one limitative and one permissive. The limitative principle, which Brouwer called the 'first act of intuitionism', is the one that limits logic by making the inference from $\neg\neg\phi$ to ϕ generally invalid. The permissive principle, which Brouwer called the 'second act of intuitionism', allows a conception of the continuum by means of free choice sequences. But in the case of arithmetic the permissive principle gets no grip and only the limitative principle is applicable. So according to the standard way of viewing matters intuitionistic arithmetic is a proper fragment of classical arithmetic. When they are laid out formally, both theories are stated in the same language, containing the usual logical constants and the standard apparatus of arithmetic (signs for primitive recursive functions such as successorhood, addition and multiplication). They have, moreover, the same mathematical axioms (defining equations for the primitive recursive functions, and all the instances of the induction schema that are stateable in the formal language). Where the two theories differ is solely in their logical rules. The classical theory (Peano Arithmetic or **PA**) admits, and the intuitionistic theory (Heyting Arithmetic or **HA**) rejects, a rule which licenses the inference from $\neg\neg\phi$ to ϕ for an arbitrary sentence ϕ . So on this view every correct proof in **HA** is a correct proof in **PA**; but the converse fails. The central case is one in which we have proved a contradiction from the assumption that $(\forall x)\phi(x)$. We

can then conclude $\neg(\forall x)\phi(x)$, which is (intuitionistically as well as classically) equivalent to $\neg\neg(\exists x)\phi(x)$. But the intuitionist will not in general accept the further step to the conclusion that $(\exists x)\phi(x)$, because he interprets that as a claim to have a method which will (in principle, at least) determine a number n such that $\phi(n)$, and merely knowing the absurdity of $(\forall x)\phi(x)$ does not in itself provide us with such a method. So on this view classical arithmetic is stated in the same language and has the same axioms as intuitionistic arithmetic, but it has more theorems because it uses a stronger logic.

But if **PA** is in this sense stronger than **HA**, it is not riskier. Suppose that we define the *negative translation* ϕ^* of an arithmetical sentence ϕ recursively as follows:

$$\begin{aligned} (\alpha = \beta)^* &=_{\text{Df}} \alpha = \beta & (\alpha \neq \beta)^* &=_{\text{Df}} \alpha \neq \beta \\ (\phi \wedge \psi)^* &=_{\text{Df}} \phi^* \wedge \psi^* & (\phi \vee \psi)^* &=_{\text{Df}} \neg(\neg\phi^* \wedge \neg\psi^*) \\ ((\forall x)\phi(x))^* &=_{\text{Df}} (\forall x)(\phi(x))^* & ((\exists x)\phi(x))^* &=_{\text{Df}} \neg(x)\neg(\phi(x))^* \\ (\neg\phi)^* &=_{\text{Df}} \neg\phi^* \end{aligned}$$

This translation is due to Gödel, who showed in 1933 that **PA** $\vdash \phi$ if and only if **HA** $\vdash \phi^*$. The significance of this is that if **PA** were contradictory there would be a sentence ϕ such that **PA** $\vdash \phi$ and **PA** $\vdash \neg\phi$, but then **HA** $\vdash \phi^*$ and **HA** $\vdash (\neg\phi)^*$, i.e. **HA** $\vdash \neg\phi^*$, so that **HA** would be contradictory too. Hence if **HA** is consistent, **PA** is also consistent: whatever else the merits of intuitionism may be, it does not buy us security from contradiction.

Now Gödel's own comment on the significance of this result was that it

shows that the system of intuitionistic arithmetic and number theory is only apparently narrower than the classical one, and in truth contains it, albeit with a somewhat deviant interpretation.¹

Gödel thus conformed to the standard view, according to which the negative translation does no more than demonstrate the relative consistency result just stated. The question I shall be concerned with here is whether it really is a 'deviant interpretation'. The original advocates of intuitionism, Brouwer and Heyting, could shrug off Gödel's result because, as Heyting put it, 'for the intuitionist this interpretation is the essential thing'.² They held that language is always in danger of outrunning the underlying meanings it is trying to express; an interpretation of one language in another is irrelevant to the consideration of these underlying meanings themselves.

It is clear that a philosophy that focusses on meaning to the exclusion of the language by means of which it is communicated is at grave risk of lapsing into solipsism. This is the objection Dummett has pressed against Brouwer's route to intuitionism. He has proposed an alternative route to the first, limitative part of the intuitionist doctrine (but not to its second, permissive part) which is concerned precisely with the relationship between language and meaning that Brouwer wished to ignore. What I shall be suggesting is that if we resist the temptation to regard the negative translation as deviant we obtain a new perspective on the relationship between classical and intuitionistic arithmetic that gives us reason to question Dummett's argument.

¹?, vol. I, p. 295.

²?, p. 18.

There is a disagreement between the intuitionist and the classicist as to the correctness of the disputed rules. But it is commonly said that the meanings of the logical connectives are given by the rules of proof that they obey: there is no more to knowing the meaning than is involved in understanding their use in inferences. If we apply this rule to all the logical rules without restriction, then we at once reach the conclusion that what the classicist means by the logical connectives is simply different from what the intuitionist means. This resolves the dispute between them, but in a rather trivial and unsatisfying manner: the classicist and the intuitionist not only do not but cannot genuinely disagree, because they are not even speaking the same language.

This is akin to saying that the Euclidean and hyperbolic geometer disagree not about the sum of the angles of a triangle but about what ‘triangle’ means. This view of geometry makes the question about the sum of the angles of a triangle one that the two geometers cannot coherently discuss. If they both, for the sake of the discussion, adopt a weaker geometry which is uncommitted on the matter, they could indeed discuss a verbally identical question, but they would now be discussing it only in relation to a third meaning of ‘triangle’ that does not coincide with what either of them understood by the term before. It seems plain that this view is too holistic by far to be at all plausible. We can insist that the word ‘triangle’ has the same meaning in the weaker geometry that it does in Euclidean and in hyperbolic geometry without rendering meanings wholly mysterious, as long as we admit that only some of the rules for using a term contribute to its meaning. But as a way of resisting holism this is so far only programmatic: it is not obvious in general which of the rules for the use of a term contribute to its meaning and which do not.

By analogy we need not simply leap to the conclusion that the classicist and the intuitionist are talking past each other. We can instead try to find a minimal theory which both accept in which the terms of the dispute make sense. But in doing so we must ensure that we do not prejudge which of the rules for each logical constant are the ones that give it its meaning.

2 Finitistic arithmetic

For definiteness let us suppose throughout that our arithmetical language includes symbols for all the primitive recursive functions. Since what is in dispute is the meanings of the logical connectives, we shall start from the basis of the system of elementary arithmetic **EA**, which has as its axioms all instances of the defining equations for all the primitive recursive functions. All true quantifier-free equations of the form $f(n) = g(n)$ and inequations of the form $f(n) \neq g(n)$ are theorems of **EA**. Since the truth values of all such sentences are mechanically decidable, we can also without difficulty apply all the propositional connectives $\neg, \vee, \wedge, \rightarrow$ to such sentences: we can, if we wish, simply define them truth-functionally in this context. The theory **EA** is thus complete and decidable.

Difficulties begin to arise only when we extend this unproblematic theory by introducing quantifiers. And of course once we do this we have fresh decisions to make about the interpretation of the propositional connectives to sentences containing quantifiers: the truth-functional account of them does not suffice to determine their meaning in cases where it is a matter of doubt whether the sentences to which they are applied have truth values themselves.

The first move beyond **EA** is to allow universal quantification over the natural numbers. Once we have the ability to generalize, we can *define* conjunction inductively by the prescription that

$$(\forall x)\phi(x) \wedge (\forall y)\psi(y) =_{\text{Df}} (\forall x)(\forall y)(\phi(x) \wedge \psi(y)).$$

I shall take it that this is still unproblematic, since it is accepted not only by intuitionists and classicists but also by the more liberal finitists.

Notice straightaway that what takes us beyond elementary arithmetic is not the addition of a new symbol — the universal quantifier — to the language of arithmetic but the addition to the axioms of all instances of the principle of mathematical induction containing the new symbol. In a language with the universal quantifier we can certainly formulate propositions that we could not formulate before, such as the commutative law for addition

$$(\forall x)(\forall y)(x + y = y + x),$$

but we are quite unable to prove even such a simple rule as this without using the principle of mathematical induction. The system of Primitive Recursive Arithmetic **PRA** is incomparably more powerful than **EA**, and the reason is that it contains as axioms all the instances of the principle of induction that are formulable in the language.

One might think that it would be a further matter of contention, once we have adopted a new logical constant, whether to adopt all the instances of induction that involve it; in practice, though, it never is. All parties to the disputes that interest us here seem ready to grant without demur that induction is valid for every instance of it that is meaningful.

Moreover, the introduction and elimination rules that govern the use of the universal quantifier within arithmetic are the same for the finitist as they are for the classicist. Only their understanding of the explanation they give of the meaning of the quantifier is different. All will agree in explaining $(\forall x)\phi(x)$ as meaning that there is a method which will produce for any number n as input a proof that $\phi(n)$ as output. But the finitist will take the assertion that there is a method as amounting to the claim that there is a finitistically acceptable proof that the method does indeed show that $\phi(n)$ in each case.

But if the introduction and elimination rules suffice to fix the meaning of a logical constant, there cannot be any disagreement between the finitist, the intuitionist and the classicist as to the meaning of the universal quantifier, since they all agree as to the introduction and elimination rules that it obeys. Indeed, if the introduction and elimination rules fix meaning, why is there any need for these informal explanations at all? A plausible answer to this question is illustrated by Arthur Prior's famous 'tonk' connective, which has the two rules

$$\frac{P}{P \text{ tonk } Q} \qquad \frac{P \text{ tonk } Q}{Q}$$

Prior argued,³ that there is no logical constant satisfying these two rules. The introduction and elimination rules suffice to fix the meaning *if any* of a purported logical constant. The role of the informal explanations is to convince us that there is a meaning there to be fixed.

³?

3 The negative fragment of intuitionistic arithmetic

The remainder of this paper may be seen as an extended test of Prior's doctrine. Let us consider first how it fares in relation to negation and implication if they are introduced next. It is at this point that we part company with the finitist, who professes not to understand these connectives when applied to universal generalizations over the natural numbers. Notice, though, that it is not the *rules* for the connectives that the finitist fails to understand: they are entirely formal and as finite as any others. What the finitist fails to understand is the informal explanation the intuitionist gives of the meanings of these connectives. If we make use of Prior's way of viewing matters, we may say that what the finitist doubts is whether there *is* a logical constant there to be denoted by the sign ' \neg '.

I shall discuss this point in more detail at the end of the article. One more observation is worth making now, though, about negation as the intuitionist understands it: for formulae in the language currently under consideration the rule of double negation elimination (from $\neg\neg\phi$ to deduce ϕ) is intuitionistically valid. This follows by induction on complexity. The atomic case amounts to no more than the observation that arithmetical equations are decidable. To prove the induction step, suppose that $\neg\neg(\forall x)\phi(x)$. Then for an arbitrary t we cannot have $\neg\phi(t)$ since that implies $\neg(\forall x)\phi(x)$, which contradicts our hypothesis. So we must have $\neg\neg\phi(t)$, from which by the induction hypothesis we obtain $\phi(t)$. Since t was arbitrary we deduce that $(\forall x)\phi(x)$.

The significance of this is that the rule of double negation elimination is the only one of the classical rules of proof that the intuitionist disputes. So within the limits of the formal language currently under consideration the classical and intuitionistic logical rules are identical. It follows that if we define \vee and \exists in the standard (classical) manner by

$$\begin{aligned}\phi \vee \psi &=_{\text{Df}} \neg(\neg\phi \wedge \neg\psi) \\ (\exists x)\phi(x) &=_{\text{Df}} \neg(\forall x)\neg\phi(x)\end{aligned}$$

we can obtain as derived rules all the classical rules of proof.

This deals with the logic of classical arithmetic; but in order to obtain the system **PA** itself we need to extend **PRA** not merely by adding negation to its logic but by adopting all the instances of induction formulable using \neg . What we then obtain is a formal system easily recognizable as first order Peano Arithmetic **PA**.

It should now be clear what the point is of approaching the matter in the manner we have adopted here: we have obtained classical arithmetic by means that are at every stage intuitionistically acceptable. In itself this is not news. What we have done is merely to repackage Gödel's negative translation, which already showed that classical arithmetic is consistent if intuitionistic arithmetic is. But the repackaging prompts a question. Is anything lost in the translation? If the doctrine presented earlier — that the meaning (if any) of a putative logical connective is determined by the rules it obeys — is correct, we can give at least a provisional answer to this question, namely that within the scope of the system we are currently considering there is no difference between the classical and intuitionistic readings of what we have set up.

But how significant is this? All we have done so far is to set up a formal system. The classicist believes that system is consistent because it is sound in

the standard interpretation; the intuitionist believes it is consistent because it is sound in the negative translation. But the consistency of a formal system is not in general all there is to its terms being meaningful. We do not show that God exists by showing that His existence is consistent. Similarly, one might say, we do not show that there are the logical constants the classicist claims there are simply by showing the consistency of a system satisfying the formal rules the classicist wants.

It is worth noting, though, that the mention of *formal* systems here does not quite hit the target. The rules for the logical constants of classical arithmetic are indeed formalizable; but the axioms are not — at least not completely. (This is what Gödel's first incompleteness theorem tells us.) And the intuitionist, on the understanding of classical practice currently under consideration, can indeed follow every step of the way in the adoption of the new axioms that result from Gödel's theorem. That is to say, the argument by which we show that the Gödel sentence $(\forall x)\phi(x)$ of **PA** is true (because $\phi(n)$ is true for each n) is intuitionistically correct. So the intuitionist should agree with the classicist's decision to adopt it as a new axiom. And similarly, of course, for other such new axioms.

The question nevertheless remains as to whether the intuitionist is understanding the classicist correctly. But this cannot be merely a familiar case of axiomatic formalism: the meaning of the terms in the language of classical arithmetic cannot be exhausted by the rules of a *formal* system.

4 The existential fragment of intuitionistic arithmetic

What we showed in the last section is that if we add the intuitionistic constants \neg and \rightarrow to the finitistically acceptable theory **PRA** and adopt all instances of mathematical induction formulable in that language, we obtain a system formally indistinguishable from **PA**. It is worth asking what happens if we add intuitionistic disjunction and the intuitionistic existential quantifier instead. We shall denote them \vee and \exists to distinguish them from their classical counterparts \vee and \exists , which we have already introduced. The explanation of $\exists x\phi(x)$ is that there is a method of finding a number n such that $\phi(n)$. The explanation of $A \vee B$ is that either there is a proof of A or there is a proof of B .

The surprising fact is that in that case too we obtain a system formally indistinguishable from **PA**. The reason is that the introduction and elimination rules for \vee and \exists are the same as those for the classical constants \vee and \exists . So all we need to do to obtain classical arithmetic from this fragment of intuitionistic arithmetic is to define classical negation recursively by the following clauses:

$$\begin{aligned} \neg(\alpha = \beta) &=_{\text{Df}} \alpha \neq \beta & \neg(\alpha \neq \beta) &=_{\text{Df}} \alpha = \beta \\ \neg(\phi \wedge \psi) &=_{\text{Df}} \neg\phi \vee \neg\psi & \neg(\phi \vee \psi) &=_{\text{Df}} \neg\phi \wedge \neg\psi \\ \neg(\forall x)\phi(x) &=_{\text{Df}} (\exists x)\neg\phi(x) & \neg(\exists x)\phi(x) &=_{\text{Df}} (\forall x)\neg\phi(x) \\ \neg\neg\phi &=_{\text{Df}} \phi \end{aligned}$$

All the rules of classical logic are then derivable.

One sometimes sees it suggested that what makes intuitionistic logic different from classical logic is that it has a stronger interpretation of the existential quantifier: the first strategy we have considered seems to bear out this view. But

the second strategy makes it seem equally plausible that the point of difference lies in the interpretation of negation. This suggests strongly that in fact what is distinctive of intuitionism is not either of the constants on its own, but the adoption of both of them together. Indeed until we have them both there is simply no way within arithmetic to say whether it is the classical or the intuitionistic constant that we are dealing with.

5 Intuitionistic arithmetic

It is not extending the logic that makes the difference, though: by adding these constants we do not extend **PA**. What makes the intuitionistic system stronger than the classical one is that once the new logical constants have been added the intuitionist accepts as axioms of **HA** all the instances of induction formulable in the new language.

Now there is a natural tendency for the classicist to suppose that he understands these intuitionistic constants perfectly well. It is, after all, a familiar fact that although classical mathematicians do not *require* constructive existence proofs, they generally prefer them. And there would be nothing to stop classical mathematicians introducing a constructive existential quantifier to indicate when there is a construction of the object in question rather than a mere proof of its existence.

But the sort of constructive quantifiers that classical mathematicians have in mind are not what the intuitionist intends at all. The reason is that if the intuitionist existential quantifier is understood as subject to the rules of classical logic then it collapses into the classical quantifier. For otherwise there would have to be a predicate $\phi(x)$ such that both $(\exists x)\phi(x)$ and $\neg(\exists x)\phi(x)$. But intuitionistically these imply $\neg(\forall x)\neg\phi(x)$ and $\neg\neg(\forall x)\neg\phi(x)$ respectively, and are therefore in direct contradiction to one another.

So someone who is a realist across the board — who believes, in other words, that classical logic applies to all discourse whatever — cannot understand the intuitionist existential quantifier as anything distinct from the classical one. It is, however, somewhat unusual to find anyone who is realist about everything: very few people are inclined towards realism about fiction, for example. A more modest realist will think that it is particular features of arithmetic that justify the application of classical logic to it. This more moderate realist might end up agreeing with the intuitionist that the rule of double negation elimination is not justified in the case of sentences involving the new quantifier: the fact that he had applied a realist logic within arithmetic up to this point would not in itself tell against such a position, since that would not automatically determine it that the realist logic is correct in the new context.

6 Dummett's dilemma

What we have arrived at, then, are two distinct conceptions on each of which — in contrast to the standard view — intuitionistic arithmetic differs from classical arithmetic not by having a weaker logic but by having a richer language in which to express instances of the induction schema. At this point, though, it is worth recalling the analogy we drew earlier with the distinction between Euclidean and hyperbolic geometry. As is well known, hyperbolic geometry can be interpreted

in Euclidean geometry in various ways: in one model the hyperbolic plane is interpreted as the open unit disk and hyperbolic lines as circular arcs cutting the unit circle at right angles; in another the hyperbolic plane is once more interpreted as the open unit disk, but hyperbolic lines are interpreted as line segments joining points on the unit circle. When the Euclidean geometer views hyperbolic geometry through the lens of either of these interpretations, it seems as if the hyperbolic plane is merely a small part of the Euclidean one. But the existence of such model does not tempt us to think that when the hyperbolic geometer talks of straight lines he really means circular arcs (or line segments). This fact seems at first to lend support to the standard view since by analogy it suggests that we should not think of the interpretation of classical arithmetic in intuitionistic arithmetic as any more than a model.

But the reason why opposition to the standard view is so natural in geometry is precisely that the words involved do not obtain their meanings solely from their roles in the geometrical theories in question. We understand what a triangle is. It is this that gives the question whether the angles of a triangle always sum to two right angles a meaning, not the purely geometrical role of the word ‘triangle’. The exponent of the standard view in logic must argue correspondingly that the logical constants used in arithmetic do not obtain their meaning from their role in arithmetical discourse alone.

The difficulty with this, though, is to see where else they could obtain their meaning from. It seems highly dubious to say that anything could determine the validity of the rule of double negation elimination in the same way that the nature of space is determined. Dummett is clear that it is not an empirical matter how things are in logical space. But to say, as Dummett does repeatedly, that the answer is to be sought in considerations in the theory of meaning scarcely seems to help, because if purely arithmetical practice does not suffice to determine the matter it is hard to see what other practice might.

Arithmetic is not, after all, actually applied within the rest of language at a level of generality that might make the difference apparent. Classical and intuitionistic arithmetic coincide not only in the equations they can prove but in the universal generalizations $(\forall x)f(x) = 0$, existential generalizations $(\exists x)f(x) = 0$ and $\forall\exists$ -formulae $(\forall x)(\exists y)f(x, y) = 0$. Among these are all the number-theoretic sentences one might ever conceivably apply in reasoning about the world. (Goldbach’s conjecture and Fermat’s last theorem are both of the first type, for instance.)

So the only way the partial intuitionist and the classicist could manifest a difference in their understanding would be by something like a hypothetical conditional. The existential intuitionist would take $(\exists x)\phi(x)$ to indicate that if he were to determine the truth values of $\phi(0)$, $\phi(1)$, etc. he would eventually find an n such that $\phi(n)$; the negative intuitionist interprets the classicist’s $(\exists x)\phi(x)$ as $\neg(\forall x)\neg\phi(x)$, from which this conditional does not follow intuitionistically. But this is to suppose that we have a secure understanding of the meaning of a hypothetical conditional whose antecedent involves an infinite task which we have no method for carrying out. It seems question-begging in the extreme to suppose such an understanding in advance of an account of quantification over the natural numbers.

7 Intuitionism and publicity

How does this leave Dummett's anti-classical argument? That was intended to suggest that the meanings the logical constants would need to have if the classical laws were to be justified are not ones they could possibly have acquired through our communal use of them. What I am arguing here, on the other hand, is that it is possible to acquire logical constants which do obey the classical laws by means that the intuitionist accepts. So either intuitionism sinks with classical arithmetic or both float together. If classical arithmetic fails the publicity test then intuitionistic negation and the intuitionistic existential quantifier do too.

It is notoriously difficult to be precise about the constraint that the publicity requirement places on our theory of meaning, and without doing that it is perhaps pointless to try to arbitrate this dilemma conclusively. I shall end, however, by indicating briefly what it is about the intuitionistic constants that might contravene Dummett's own strictures.

Negation first. My suggestion is that the meaning of intuitionistic negation cannot be manifested in the manner Dummett requires. The reason is that by claiming to understand negation the intuitionist quantifies over, and therefore on his own principles lays claim to a conception of, the totality of all (canonical) proofs of arithmetical sentences, but any explanation of our grasp of this totality seems inevitably to involve a conditional whose unreality makes its meaning as opaque as any appealed to by the realist. For the intuitionist, let us recall, explains $\neg\phi$ as meaning that there is a method which converts any proof of ϕ into a proof of the explicit contradiction $0 = 1$. The difficulty with this explanation for the finitist is that it generalizes over proofs. Since the finitist understands universal generalization over natural numbers, he certainly understands generalization over any recursively enumerable set of strings. He can therefore understand generalization over the proofs of a formal theory, since they are recursively enumerable as the range of some primitive recursive function. Thus if T is a formal theory, we may consider a connective \neg_T explained so that $\neg_T\phi$ means that there is a method for converting any proof of ϕ in T into a proof that $0 = 1$ in T .

But if this is not to be impredicative, the theory T cannot involve implication. If, for instance, T_0 is **PRA**, we obtain a connective \neg_0 explained so that $\neg_0\phi$ means that any proof of ϕ in **PRA** is convertible into a proof that $0 = 1$. But we can then extend **PRA** by adding to it all the instances of induction involving the new connective \neg_0 to obtain a larger theory T_1 . We can then introduce a connective \neg_1 defined in terms of T_1 . This leads us to a theory T_2 containing all the instances of induction involving \neg_1 . And so on.

So far, of course, this iterative procedure is finitistically legitimate, even if it is iterated into the transfinite. In general, we define $T_{\alpha+1}$ to be T_α with all instances of induction involving \neg_α added; and T_λ to be the union of the T_α for $\alpha < \lambda$ whenever λ is a limit ordinal. But eventually (when $\alpha = \omega^\omega$) the procedure of explaining \neg_α is one that the finitist cannot understand (because he can make no finitistic sense of the ordinal involved).

The best the finitist can do, therefore, is to form a hierarchy of implication connectives. The intuitionist, on the other hand, claims to understand the original explanation once and for all as a generalization over *all* proofs whatever, not those restricted to some particular formal theory. It is the intuitionist's claim to be able to understand this generalization that gives the negative fragment of

intuitionistic arithmetic its power to emulate classical arithmetic.

Similar remarks apply to the intuitionistic existential quantifier. The intuitionist interprets $(\exists x)\phi(x)$ as meaning that there is a method for finding a number of which ϕ holds. The difficulty with understanding this comes when we try to say what is to count as an intuitionistically acceptable method. Suppose that the method for finding the number is encoded in an algorithm. In order for it to be of any use to use we need a proof that this algorithm will terminate. But this problem is of just the same order of complexity as the problem of proving arithmetical theorems on which our understanding of intuitionistic negation hinged. For any formal theory T we can of course define a quantifier \exists_T in such a way that $(\exists_T x)\phi(x)$ means there is an algorithm for producing a number satisfying ϕ which can be proved in T to terminate. But the general notion is as resistant to non-circular explanation as negation was.

8 Conclusion

My aim has been to sketch a dilemma for Dummett. If he draws the constraints on what is to constitute manifestation of the meaning of a logical constant too narrowly, his position will collapse into finitism, with disastrous consequences for arithmetic as generally understood. If he draws them more generously, however, so as to make our grasp of intuitionistic negation or the intuitionistic existential quantifier plausible, he renders legitimate the practice of the negative intuitionist and the existential intuitionist respectively. But his challenge in that case is to specify — without appeal to hypothetical conditionals referring in the antecedent to infinite tasks — what is the manifestable difference between their practice and that of the classicist his argument was originally intended to criticize.

References

- Gödel, Kurt, *Collected Works* (Oxford University Press, 1986–2003)
- Heyting, Arend, *Mathematische Grundlagenforschung. Intuitionismus. Beweistheorie* (Berlin: Springer, 1934)
- Prior, A. N., ‘Conjunction and Contonktion Revisited’, *Analysis*, 24 (1964), 191–5 (ArticleType: primary_article / full publication date: jun., 1964 / copyright © 1964 the analysis committee)