Part IB Logic

Class 1: Metatheory of propositional calculus I

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3pm Tuesday 28th January 2014
Mill Lane Room 10

Do not hand in your solutions in advance: write them out and bring them with you to the class. For explanations of the terminology used, consult the `Metatheory` lecture notes at `http://people.ds.cam.ac.uk/tecb2/teaching.shtml`.

Induction

1. Prove by induction that $n^3 - n$ is divisible by n for every natural number n.

2. Prove by induction that $n^2 - 1$ is divisible by 8 for odd n.

3. The *least element principle* says that every non-empty set of natural numbers has a least element.

   (a) Prove the least element principle from the principle of induction.

   (b) Prove the principle of induction from the least element principle. What properties of natural numbers are you assuming in your proof?

Complexity

Use '$\Box$' to stand for any of the binary connectives of TFL. The *complexity* comp($\mathcal{A}$) of a sentence $\mathcal{A}$ of TFL is defined by

$$\begin{aligned} \text{comp}(\mathcal{A}) &= 0 \text{ if } \mathcal{A} \text{ is atomic} \\ \text{comp}(\mathcal{A}\Box\mathcal{B}) &= \max(\text{comp}(\mathcal{A}), \text{comp}(\mathcal{B})) + 1 \\ \text{comp}(\neg\mathcal{A}) &= \text{comp}(\mathcal{A}) + 1 \end{aligned}$$

4. (a) Find the complexity of:

      i. $(P \wedge (\neg Q \vee R))$;

      ii. $(P \to (Q \to (R \to S)))$

      iii. $((P \to Q) \to (R \to S))$

   (b) Let length($\mathcal{A}$) be the number of occurrences of connectives in $\mathcal{A}$. Show that comp($\mathcal{A}$) $\leqslant$ length($\mathcal{A}$). Give examples to show that both equality and strict inequality may occur here.

5. Show by induction on complexity that every sentence of TFL has an even number of brackets.

6. Show by induction that if length($\mathcal{A}$) $= n$ then $\mathcal{A}$ has at most $2n + 1$ subsentences.

7. Let atom($\mathcal{A}$) be the number of occurences of atomic sentences in $\mathcal{A}$. Show that length($\mathcal{A}$) + atom($\mathcal{A}$) $\leqslant 2^{\text{comp}(\mathcal{A})+1} - 1$.

Normal forms

8. Read Chapter 3 of `Metatheory`. Then consider the following sentences:

   (a) $\neg(A \leftrightarrow B)$

   (b) $(\neg(A \to B) \wedge (A \to C))$

   (c) $((\neg(A \wedge \neg B) \to C) \wedge \neg(A \wedge D))$

   For each sentence use both algorithms (by truth table, and by substitution) to write down sentences in DNF that are tautologically equivalent to these sentences.

Expressive adequacy

9. Which of the following lists of connectives are expressively adequate? Give reasons.

   (a) $\neg, \vee, \wedge$.

   (b) $\neg, \vee$.

   (c) $\vee, \wedge, \leftrightarrow, \to$.

   (d) $\uparrow$.

   (e) $\downarrow$.

   (f) $\neg, \leftrightarrow$.

   (For the definitions of $\uparrow$ and $\downarrow$ consult chapter 4 of `Metatheory`.)

10. Which binary connectives are expressively adequate on their own?