Part IB Logic

Class 3: Theories

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3pm Tuesday 11th February 2014

1. Are the following statements true or false? In each case, justify your answer.

   (a) If there is an interpretation that satisfies an axiom set Σ but not some sentence ϕ, then ϕ is independent of Σ.

   (b) If there is some sentence ϕ such that theory $\Theta \vdash \phi$ and $\Theta \vdash \neg\phi$, then Θ is negation incomplete.

   (c) If theory Θ has a model that satisfies some sentence ϕ and another model that satisfies $\neg\phi$, then ϕ is independent of Θ.

   (d) If the removal of an axiom α from theory Θ's axiom set changes Θ from a negation complete theory to a negation incomplete theory, then α is independent of Θ.

   (e) If theory $\Theta \models \phi$ but $\Theta \nvdash \phi$, then Θ's deductive system is negation incomplete.

   (f) If theory Θ has a model, then it is consistent.

   (g) If theory Θ is consistent, then it has a model.

2. Incidence geometry has the following 3 axioms:

   (1) For every point P and for every point Q not equal to P there exists a unique line l incident with P and Q.

   (2) For every line l there exist at least two distinct points incident with l.

   (3) There exist three distinct points with the property that no line is incident with all three of them.

   Show that the following are theorems of incidence geometry:

   (a) If l and m are distinct lines that are not parallel, then l and m have a unique point in common.

   (b) For every line there is at least one point not lying on it.

   (c) For every point P there exist at least two distinct lines that are incident with P.

   (See Chapter 2 of Greenberg's *Euclidean and Non-Euclidean Geometries* for more on incidence geometry.)

3. Find out what is meant by the Beltrami-Klein disk model. (*Hint* Google it.)

   (a) How should we reinterpret the primitives 'point', 'line' and 'is incident with' for the Beltrami-Klein disk model?

   (b) Reinterpret the axioms of incidence geometry accordingly and check that so interpreted they are true.

   (c) The *hyperbolic axiom* is:

> For every line *l* and every point *P* that does not lie on *l*, there is more than one line *m* that can be drawn through *P* that is parallel to *l*.

Reinterpret this axiom for the Beltrami-Klein disk model and check that, so interpreted, it is a truth of Euclidean geometry.

4. Find out what is meant by the Poincaré disk model. (*Hint* Google it.)

   (a) How should we reinterpret the primitives 'point', 'line' and 'is incident with' for the Poincaré disk model?

   (b) Reinterpret the axioms of incidence geometry accordingly and check that so interpreted they are true.

   (c) Reinterpret the hyperbolic axiom for the Poincaré model and check that, so interpreted, it is a truth of Euclidean geometry.

   (d) Does the Poincaré disk model have any advantages over the Beltrami-Klein disk model. (*Hint* Think about angles.)

   (See Chapter 7 of Greenberg's *Euclidean and Non-Euclidean Geometries* for more on hyperbolic geometry.)