Part IB Logic
Class 4: Intuitionistic Logic

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Intuitionistic propositional logic IPL is like TFL except that the rule TND is omitted. We shall write $\vdash_{\mathrm{I}}$ for provability in IPL and $\vdash_{\mathrm{C}}$ for provability in TFL.

1. (a) Prove $\vdash_{\mathrm{I}} \neg\neg(\mathcal{A} \vee \neg\mathcal{A})$.
   (b) Prove $\vdash_{\mathrm{I}} \neg(\mathcal{A} \wedge \neg\mathcal{A})$.

2. Prove that the following are *equivalent*, against the background of intuitionistic logic:

   (i) The Law of Excluded Middle (i.e. that any instance of $\mathcal{A} \vee \neg\mathcal{A}$ is an *Axiom*)
   (ii) Unrestricted instance of the rule TND (as defined in `forallx`)
   (iii) Unrestricted instance of the rule DNE (as defined in `forallx`)

   Prove that $\neg(\mathcal{A} \vee \neg\mathcal{A})$ is a schematic *logical contradiction*, for intuitionists.

3. The *Gödel translation* of a formula involves sprinkling that formula with additional negation signs, according to the following recursive definition:

$$
\begin{aligned}
\mathcal{A}^{\mathrm{g}} &= \neg\neg\mathcal{A}, \text{ if } \mathcal{A} \text{ is atomic}\\
(\mathcal{A} \wedge \mathcal{B})^{\mathrm{g}} &= (\mathcal{A}^{\mathrm{g}} \wedge \mathcal{B}^{\mathrm{g}})\\
(\mathcal{A} \vee \mathcal{B})^{\mathrm{g}} &= \neg(\neg\mathcal{A}^{\mathrm{g}} \wedge \neg\mathcal{B}^{\mathrm{g}})\\
(\mathcal{A} \to \mathcal{B})^{\mathrm{g}} &= (\mathcal{A}^{\mathrm{g}} \to \mathcal{B}^{\mathrm{g}})\\
(\neg\mathcal{A})^{\mathrm{g}} &= \neg\mathcal{A}^{\mathrm{g}}
\end{aligned}
$$

   Prove that the Gödel translation has the following interesting properties:

   (a) $\vdash_{\mathrm{C}} (\mathcal{A} \leftrightarrow \mathcal{A}^{\mathrm{g}})$
   (b) $\vdash_{\mathrm{I}} (\mathcal{A}^{\mathrm{g}} \leftrightarrow \neg\neg\mathcal{A}^{\mathrm{g}})$
   (c) If $\vdash_{\mathrm{C}} \mathcal{A}$, then $\vdash_{\mathrm{I}} \mathcal{A}^{\mathrm{g}}$

   Hence, where Γ^{g} are the Gödel translation of every sentence among Γ, prove the following:

   (a) $\Gamma \vdash_{\mathrm{C}} \mathcal{A}$ iff $\Gamma^{\mathrm{g}} \vdash_{\mathrm{I}} \mathcal{A}^{\mathrm{g}}$
   (b) $\Gamma \vdash_{\mathrm{C}} \bot$ iff $\Gamma \vdash_{\mathrm{I}} \bot$