

Model answers:

1. (a)
 - (i) An argument is (informally) **valid** if and only if there is no possible way for the premises to be true and the conclusion to be false.
 - (ii) An argument is **tautologically valid** if and only if there is no valuation of the atoms involved in the premises and conclusion that makes the premises true and the conclusion false.
 - (iii) A wff of PL is a **tautology** iff it takes the valuation true on every valuation of its atoms.

(b) ' \therefore ' is an inference marker in the object language PL i.e. it connects wffs of PL; ' \models ' is a symbol of the metalanguage indicating that a set of wffs semantically entails another wff; ' \supset ' is an operator in the object language taking pairs of wffs to wffs: $p \supset q$ is true if and only if q is true or p is false.

(c) No. Any valuation that makes a set X of premises true makes every subset of X true as well. So if a valuation makes X true and C false then it makes every subset of X true whilst making C false. So if Y is a subset of X then the inference from Y to C is invalid. So there are no X , Y and C such that Y is a subset of X , the argument from Y to C is valid, and the argument from X to C is invalid.

(d) **Sound**: if any tree with trunk $P_1, P_2, \dots, P_n, \neg C$ closes then the argument from P_1, P_2, \dots, P_n to C is tautologically valid. **Complete**: if the argument from P_1, P_2, \dots, P_n to C is tautologically valid then any tree with trunk $P_1, P_2, \dots, P_n, \neg C$ closes.

(e) A set of connectives is **expressively adequate** if a language containing just these connectives is rich enough to express all truth-functions of the atomic sentences of the language. Proof that $\{\neg, \vee\}$ is expressively adequate: bookwork (see IFL: 11.7 and 11.9).

2. (a)
 - (i) A **term** of $QL=$ is an individual constant or individual variable.
 - (ii) An **operator** of $QL=$ is a connective or quantifier.
 - (iii) The **scope** of an operator is the wff (or wffs) that occur at the last step of a construction tree before that operator is introduced.
 - (iv) A **q-valuation** specified over a vocabulary V : specifies a non-empty domain D ; assigns to any constant in V some object in D as its q-value; assigns to any n -place predicate in V , where $n > 0$, a (possibly empty) set of n -tuples of objects from D .
 - (v) An **extended q-valuation** defined over vocabulary V is a q-valuation of V augmented by the assignment of objects as q-values to one or more variables.

- (vi) An argument is **q-valid** if and only if there is no q-valuation of the vocabulary V involved in its premises and conclusion that makes the premises true and the conclusion false.

(b) For any wff $C(\dots v \dots v \dots)$ with variable v free, q makes $\forall v C(\dots v \dots v \dots)$ true iff every v -variant of q makes $C(\dots v \dots v \dots)$ true. For any wff $C(\dots v \dots v \dots)$ with variable v free, q makes $\exists v C(\dots v \dots v \dots)$ true iff *some* v -variant of q makes $C(\dots v \dots v \dots)$ true.

Suppose that q makes $\neg \exists x Fx$ true. Then no x -variant of q makes Fx true. So every x -variant of q makes Fx false. So every x -variant of q makes $\neg Fx$ true. So q makes $\forall x \neg Fx$ true.

3. (a)
 (i) Reflexivity: $\forall x Rxx$
 (ii) Symmetry: $\forall x \forall y (Rxy \equiv Ryx)$
 (iii) Transitivity: $\forall x \forall y \forall z ((Rxy \wedge Ryz) \supset Rxz)$

(b) see scanned attachments

- (c)
 (i) None
 (ii) $y = \text{Batman}$ (on any domain including Batman and somebody else)
 (iii) $\neg x = x$ (the empty relation)
 (iv) $x = \text{Batman}$ (on any domain including Batman and somebody else)

4. (a)
 (i) $\Pr(2S) = \frac{1}{4}$
 (ii) $\Pr(1S \mid \neg 2S)$
 $= \Pr(1S \wedge \neg 2S) / \Pr(\neg 2S)$
 $= (13/52 \times 39/51) / (39/52)$
 $= 13/51$
 (iii) $\Pr(2C \mid 1H)$
 $= \Pr(2C \wedge 1H) / \Pr(1H)$
 $= \Pr(13/52 \times 13/51) / (13/52)$
 $= 13/51$
 (iv) $\Pr(2D \mid 1D)$
 $= \Pr(1D \wedge 2D) / \Pr(1D)$
 $= (13/52 \times 12/51) / (13/52)$
 $= 4/17$

$$\begin{aligned}
& \text{(v) } \Pr(2D \mid 1KD) \\
&= \Pr(1KD \wedge 2D) / \Pr(1KD) \\
&= (1/52 \times 12/51) / (1/52) \\
&= 4/17
\end{aligned}$$

(b) Neither is right but the prosecution is closer to the truth. It all depends on (i) how likely it is that Billy was at the scene (ii) how likely it was that somebody else with Billy's DNA was at the scene.

B = Billy was at the scene
D = Billy's DNA was at the scene

We have:

$$\begin{aligned}
\Pr(B \mid D) &= \Pr(B) \Pr(D \mid B) / [\Pr(B) \Pr(D \mid B) + \Pr(\neg B) \Pr(D \mid \neg B)] \\
&= \Pr(B) / [\Pr(B) + \Pr(D \mid \neg B) (1 - \Pr(B))]
\end{aligned}$$

--here making the assumption that if Billy was at the scene then his DNA would be there too. The crucial quantities are then (i) $\Pr(B)$ and (ii) $\Pr(D \mid B)$.

The fact that the prosecution cites makes it reasonable to set $\Pr(D \mid B)$ very low e.g. at about 0.01% if we know only that only one person was at the scene at the time in question. If our prior probability of Billy's guilt is not extraordinarily low (so low that he would never have come to trial in the first place) then $\Pr(B \mid D)$ is close to 1 and the DNA evidence should convict him.

If on the other hand we have some reason to suspect that somebody else with Billy's DNA was around at the time, $\Pr(D \mid \neg B)$ may be quite high; and as $\Pr(D \mid \neg B)$ approaches 1 we find that the DNA evidence does almost nothing to incriminate Billy. Note finally that even if $\Pr(D \mid \neg B) = 1$ and the DNA evidence is completely ineffectual, still that does nothing to vindicate the defence lawyer's absurd claim that it effectively *exonerates* Billy.